Second order viscous corrections to the harmonic spectrum in heavy ion collisions

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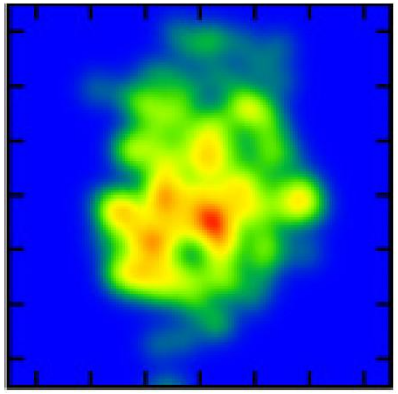
In collaboration with Derek Teaney

Ingredients of hydrodynamic simulations for Heavy-ion collisions.

- Initial state: initial condition of hydro. evolution



- Hydro. EOM: conservation of energy-momentum (and charge)

$$\partial_\mu T^{\mu\nu} = 0 \quad \& \quad (\partial_\mu n_B^\mu = 0) \longrightarrow u^\mu, e, \text{and gradients of these fields}$$

- Freeze-out: generate particle spectrum from hydro.,

$$E\frac{dN}{d\mathbf{p}^3} = \frac{\nu}{(2\pi)^3}\int_\Sigma p\cdot\sigma f(x,\mathbf{p})$$

- ...

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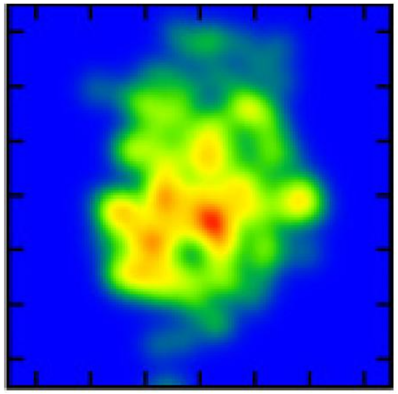

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Properties of initial state and medium can be estimated from predictions.

Viscous hydrodynamics and dissipative effects of medium

In hydro. model, dissipative corrections of higher order in the gradient expansion:

- EOM

$$T^{\mu\nu} = eu^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + \pi^{\mu\nu}$$

1. Bulk viscosity $\zeta = 0$, so no bulk tensor.
2. Due to causality problem, 2nd order EOM ($\sim \nabla^2$) has to be considered.
3. Transport coefficient, η/s as input parameters for hydro.

- freeze-out corrections to the particle spectrum

$$E\frac{dN}{d\mathbf{p}^3} = \frac{\nu}{(2\pi)^3}\int_\Sigma p\cdot\sigma[n(x,\mathbf{p}) + \delta f(x,\mathbf{p})]$$

where,

$$\delta f(x,\mathbf{p}) = \frac{n(1\pm n)}{2(e+\mathcal{P})T^2}p^\mu p^\nu \pi_{\mu\nu}$$

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But $\delta f \sim pp\pi$ is not self-consistent in the hydro. calculations.

Consistency of δf – continuity of $T^{\mu\nu}$ at freeze-out

Kinetic theory $\leftrightarrow$ $p \cdot \partial f = -\mathcal{C}[f]$

- Form of $f(x, \mathbf{p})$ is constrained as a result of consistency,

$$T_0^{\mu\nu} + \pi^{\mu\nu} = g \int \frac{d^3\mathbf{p}}{(2\pi)^3 E} p^\mu p^\nu (n(x,\mathbf{p}) + \delta f(x,\mathbf{p})),$$

- Freeze-out ...

$$E\frac{d^3N}{d\mathbf{p}^3} = \frac{g}{(2\pi)^3} \int_\Sigma p^\mu d\sigma_\mu (n(x,\mathbf{p}) + \delta f(x,\mathbf{p}))$$

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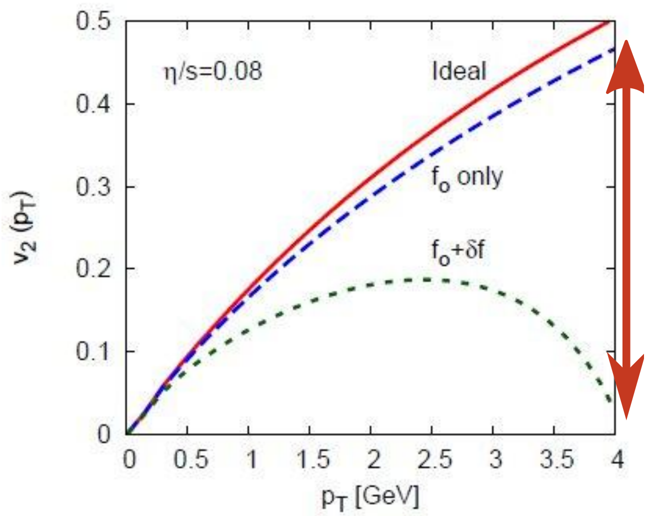

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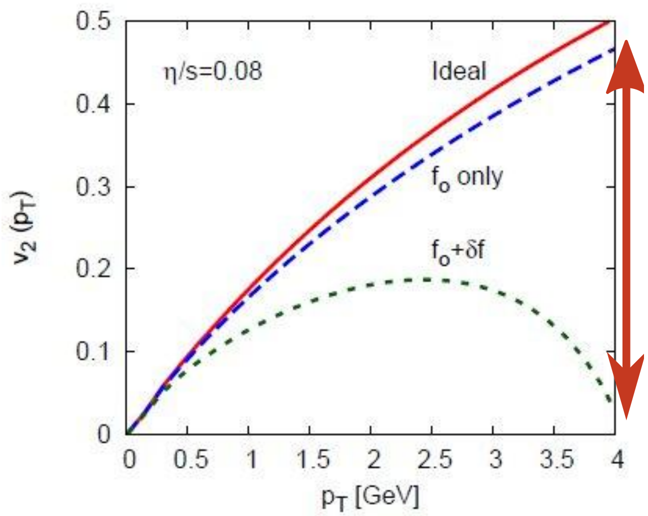


- Most damping of flow from δf at freeze-out.
- Hydro EOM : 2nd order viscous corrections.
- Freeze-out : $f(x, \mathbf{p})$(to "1st order"),

$$f(x, \mathbf{p}) = \underbrace{n(x, \mathbf{p})}_{\text{ideal dist.}} + \underbrace{\delta f_1}_{\propto p^\mu p^\nu \pi_{\mu\nu}} + \underbrace{\delta f_2}_{?} + \ldots.$$

Determine δf for 1st order viscous hydro – (Navier-Stokes hydro.)

Stress tensor with 1st order viscous corrections,

$$\pi^{\mu\nu} = -\eta \underbrace{2\nabla^{\langle\mu} u^{\nu\rangle}}_{\sigma^{\mu\nu}}, \qquad \begin{cases} \nabla^\mu = \Delta^{\mu\nu}\partial_\nu \sim \partial^i \\ D = u^\mu \partial_\mu \sim \partial_t \\ \langle \ldots \rangle \text{symmetric, traceless, transverse} \end{cases}$$

1. Transport equation for $f(x, \mathbf{p}) = n(x, \mathbf{p}) + \delta f_1(x, \mathbf{p}) + o(\nabla)^2$,

$$p^\mu \partial_\mu n(x, \mathbf{p}) = -\mathcal{C}[f(x, \mathbf{p})] \equiv \frac{(p \cdot u)\delta f_1}{\tau_R} = -\frac{T^2 \delta f_1}{\tilde{C}_R}$$

- 'tilde' stands for dimensionless quantity, e.g., $\tilde{p}^\mu = p^\mu / T$.
- Relaxation time[1] $\tau_R \propto (\tilde{p} \cdot u)^{1-\alpha}$, $\rightarrow \tilde{C}_R = -\tilde{\tau}_R / \tilde{p} \cdot u = c_o (\tilde{p} \cdot u)^{-\alpha}$.

$$\underbrace{\alpha = 0}_{\text{quadratic ansatz}} \qquad \Longleftarrow \qquad \alpha \qquad \Longrightarrow \qquad \underbrace{\alpha = 1}_{\text{linear ansatz}}$$

$\delta f_1 \propto p_T^2$	$\delta f_1 \propto p_T$
$\tau_R \propto E_p$	$\tau_R \propto const.$
collisional E-loss	radiative E-loss

[1] K. Dusling, G. Moore and D. Teaney

2. 0th order hydro eom,

$$D\varepsilon = -(\varepsilon + P)\nabla \cdot u$$
$$Du^\nu = -\frac{\nabla^\nu P}{\varepsilon + P} = -\nabla^\nu \ln T$$

so,

$$p \cdot \partial n(x, \mathbf{p}) = \frac{n(1 \pm n)}{2T} p^\mu p^\nu \sigma_{\mu\nu}.$$

3. Then from transport equation,

$$\rightarrow \qquad \delta f_1 = -\frac{n' \tilde{C}_R}{2T} \tilde{p}^\mu \tilde{p}^\nu \sigma_{\mu\nu}$$

4. Matching conditions: $\pi^{\mu\nu} = -\eta\sigma^{\mu\nu} = \int_p p^\mu p^\nu \delta f_1$ and $u_\mu T^{\mu\nu} = eu^\nu$

$$\rightarrow \quad \eta = \frac{\nu T^3}{15} \int \frac{d^3\tilde{p}}{(2\pi)^3 \tilde{E}} [n(1 \pm n)\tilde{C}_R](\tilde{p} \cdot u)^4 \quad \xrightarrow{\alpha=0} c_o = \eta/s$$

Simple summary of the 4 steps

- δf is expanded into spatial gradients order by order
- Hydro. EOM converts time derivatives into space gradients.
- $p \cdot \partial$ in transport equ. gives rise to $\nabla + \nabla^2 + \ldots$.
- Correspondence between $f(x, \mathbf{p})$ and hydro $T^{\mu\nu} \to c_0$.
- η/s is the only input parameter (also for 2nd order).
- Conformal symmetry has been taken into account, so $m = 0$.

2nd order viscous hydro. – (BRSSS hydro.)

Stress tensor with second order viscous corrections(BRSSS) [2]

$$
\begin{aligned}
\pi^{\mu\nu} =& -\eta\sigma^{\mu\nu} \\
&+ \eta\tau_\pi \left[{}^{\langle} D\sigma^{\mu\nu\rangle} + \frac{1}{d-1}\sigma^{\mu\nu}\nabla\cdot u \right] \\
&+ \lambda_1 \sigma_\lambda^{\langle\mu}\sigma^{\nu\rangle\lambda} + \lambda_2 \sigma_\lambda^{\langle\mu}\Omega^{\nu\rangle\lambda} + \lambda_3 \Omega_\lambda^{\langle\mu}\Omega^{\nu\rangle\lambda},
\end{aligned}
$$

- Conformal symmetry assumed.
- Structure $\Omega^{\mu\nu} = \nabla^\mu u^\nu - \nabla^\nu u^\mu$, antisymmetric.
- 2nd order transport coefficients: τ_π, λ_1, λ_2 and λ_3.

[2] R. Baier, P. Romatschke, D. Son, A. Starinets and M. Stephanov

Determine the corresponding form of $\delta f_2(x, \mathbf{p})$.

1. For $f(x, \mathbf{p}) = n(x, \mathbf{p}) + \delta f_1(x, \mathbf{p}) + \delta f_2(x, \mathbf{p}) + o(\nabla)^3$,

$$p^\mu \partial_\mu [n(x, \mathbf{p}) + \delta^{(1)} f(x, \mathbf{p})] = \frac{(p \cdot u)\delta^{(2)} f(x, \mathbf{p})}{\tau_R}$$

2. 1st order hydro eom,

$$\begin{aligned}
&\partial_\mu T^{\mu\nu} = 0 \\
&\to D\varepsilon = -(\varepsilon + P)\nabla \cdot u + \frac{\eta}{4}\sigma_{\mu\nu}\sigma^{\mu\nu} \\
&\to Du_\alpha = -\frac{\nabla_\alpha P}{\varepsilon + P} + \frac{u_\mu D\pi^{\mu\nu}\Delta_{\alpha\nu}}{\varepsilon + P} - \frac{\Delta_{\alpha\nu}\nabla_\mu \pi^{\mu\nu}}{\varepsilon + P}
\end{aligned}$$

3. Identify the terms in transport equ. w.r.t. 2nd order gradients,

$$\delta f_2 = \frac{\eta}{s}\frac{n'\tilde{C}_R}{T^2}\tilde{p}\cdot u\left[\tilde{p}\cdot q - \frac{\tilde{p}\cdot u}{4(d-1)}\sigma^2\right] + \frac{[n'\tilde{C}_R]'\tilde{C}_R}{4T^2}(\tilde{p}^\mu\tilde{p}^\nu\sigma_{\mu\nu})^2$$
$$- \frac{3[n'\tilde{C}_R]\tilde{C}_R}{2T^2}\tilde{p}^\mu\tilde{p}^\nu\sigma_{\mu\nu}\left[\tilde{p}\cdot u\frac{\nabla\cdot u}{d-1} + \tilde{p}^\alpha\nabla_\alpha lnT\right]$$
$$+ \frac{[n'\tilde{C}_R]\tilde{C}_R}{2T^2}\tilde{p}^\mu\tilde{p}^\nu\left[-\tilde{p}\cdot uD\sigma_{\mu\nu} + \tilde{p}^\alpha\nabla_\alpha\sigma_{\mu\nu}\right]$$

4. Compare with,

$$\pi^{(2)\mu\nu} = \eta\tau_\pi\left[D\sigma^{\langle\mu\nu\rangle} + \frac{1}{d-1}\sigma^{\mu\nu}\nabla\cdot u\right]$$
$$+ \lambda_1\sigma_\lambda^{\langle\mu}\sigma^{\nu\rangle\lambda} + \lambda_2\sigma_\lambda^{\langle\mu}\Omega^{\nu\rangle\lambda} + \lambda_3\Omega_\lambda^{\langle\mu}\Omega^{\nu\rangle\lambda},$$

- The matching condition $\pi^{(2)\mu\nu} = \int_p p^\mu p^\nu \delta f_{(2)}$
- u^μ with 2nd order corrections.

Straightforward from δf_2,

$$\pi^{\rho\sigma(2)}/(T^2\nu) = \sigma^{\langle\rho\lambda}\sigma^{\sigma\rangle}_{\lambda}\left[\frac{2B_2}{15(d+3)} - \frac{B_3}{15}\right] - \frac{2B_3}{15}\left\{\sigma^{\langle\rho\lambda}\Omega^{\sigma\rangle}_{\lambda} - \frac{1}{2}\left[{}^{\langle}D\sigma^{\rho\sigma\rangle} + \frac{\sigma^{\rho\sigma}}{d-1}\nabla\cdot u\right]\right\}$$

where B's are constants and depend on c_o (or η/s) and α

$$B_1 = \int_{\tilde{p}}[n'\tilde{C}_R](\tilde{p}\cdot u)^4 \sim \int_{\tilde{p}}[n'\tilde{C}_R]\tilde{p}^\rho\tilde{p}^\sigma\tilde{p}^\alpha\tilde{p}^\beta,$$

$$B_2 = \int_{\tilde{p}}[n'\tilde{C}_R]'\tilde{C}_R(\tilde{p}\cdot u)^6 \sim \int_{\tilde{p}}[n'\tilde{C}_R]'\tilde{C}_R\tilde{p}^\rho\tilde{p}^\sigma\tilde{p}^\alpha\tilde{p}^\beta\tilde{p}^\mu\tilde{p}^\nu,$$

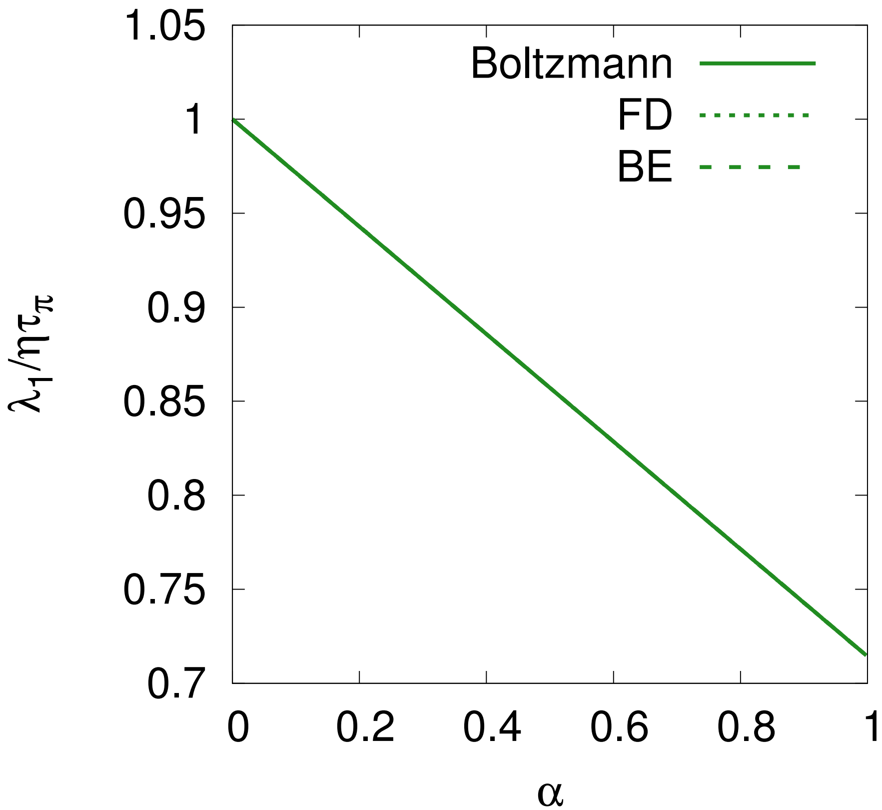
$$B_3 = -\int_{\tilde{p}}[n'\tilde{C}_R]\tilde{C}_R(\tilde{p}\cdot u)^5 \sim \int_{\tilde{p}}[n'\tilde{C}_R]\tilde{C}_R\tilde{p}^\rho\tilde{p}^\sigma\tilde{p}^\alpha\tilde{p}^\beta\tilde{p}^\mu.$$

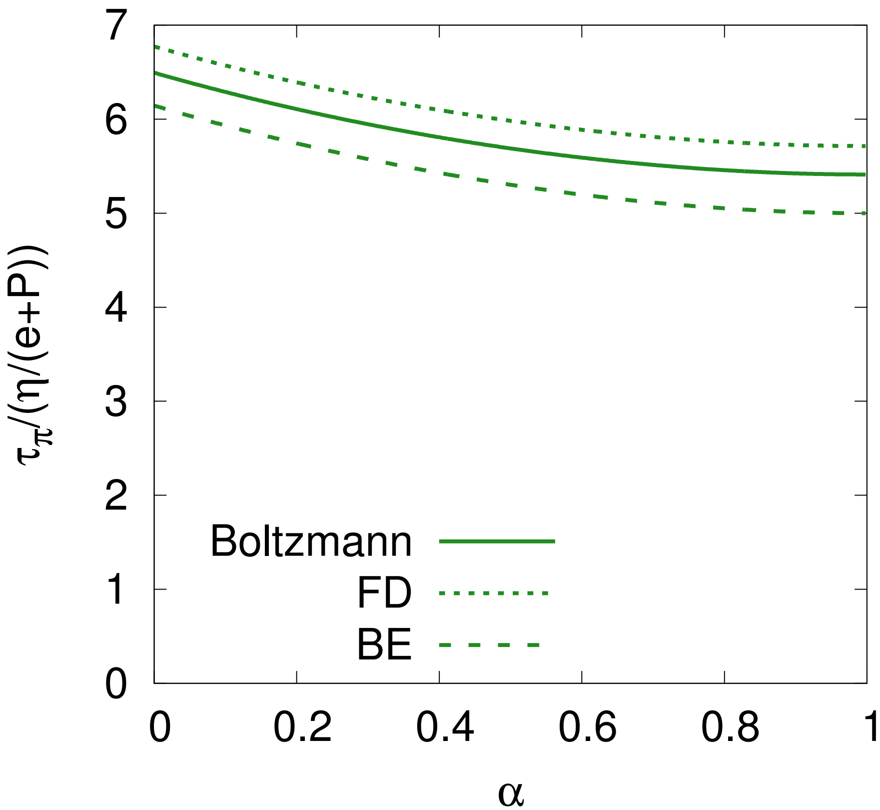
so,

$$\lambda_1 = \left[\frac{2B_2}{15(d+3)} - \frac{B_3}{15}\right]\nu T^2, \quad \lambda_2 = -2\eta\tau_\pi$$

$$\lambda_3 = 0, \qquad \eta\tau_\pi = B_3\nu T^2/15$$

With conformal EOS,





$\delta f_{(2)}$ and harmonic flow v_n

Linear harmonic flow response coefficient: w_n/ϵ_n

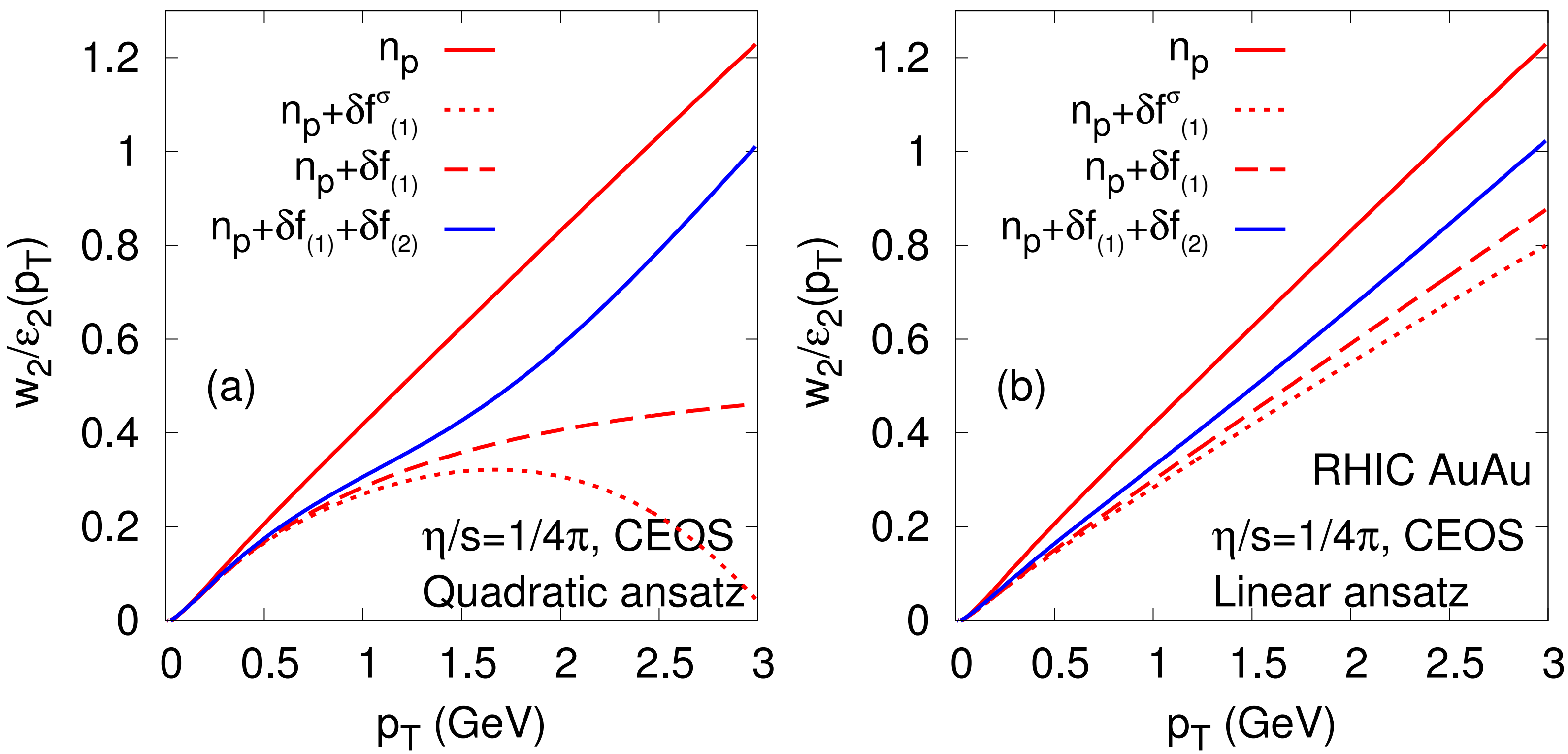
$$v_n = \sqrt{\left(\frac{w_n}{\epsilon_n}\right)^2 \epsilon_n^2}$$

The dependence of v_n on freeze-out δf.

- n_p : n_p as freeze-out distribution function.
- $n_p + \delta f^{\sigma}_{(1)}$: $\delta f \propto pp\sigma$.
- $n_p + \delta f_{(1)}$: $\delta f \propto pp\pi$.
- $n_p + \delta f_{(1)} + \delta f_{(2)}$: 2nd order δf.

The dependence of v_n on α.

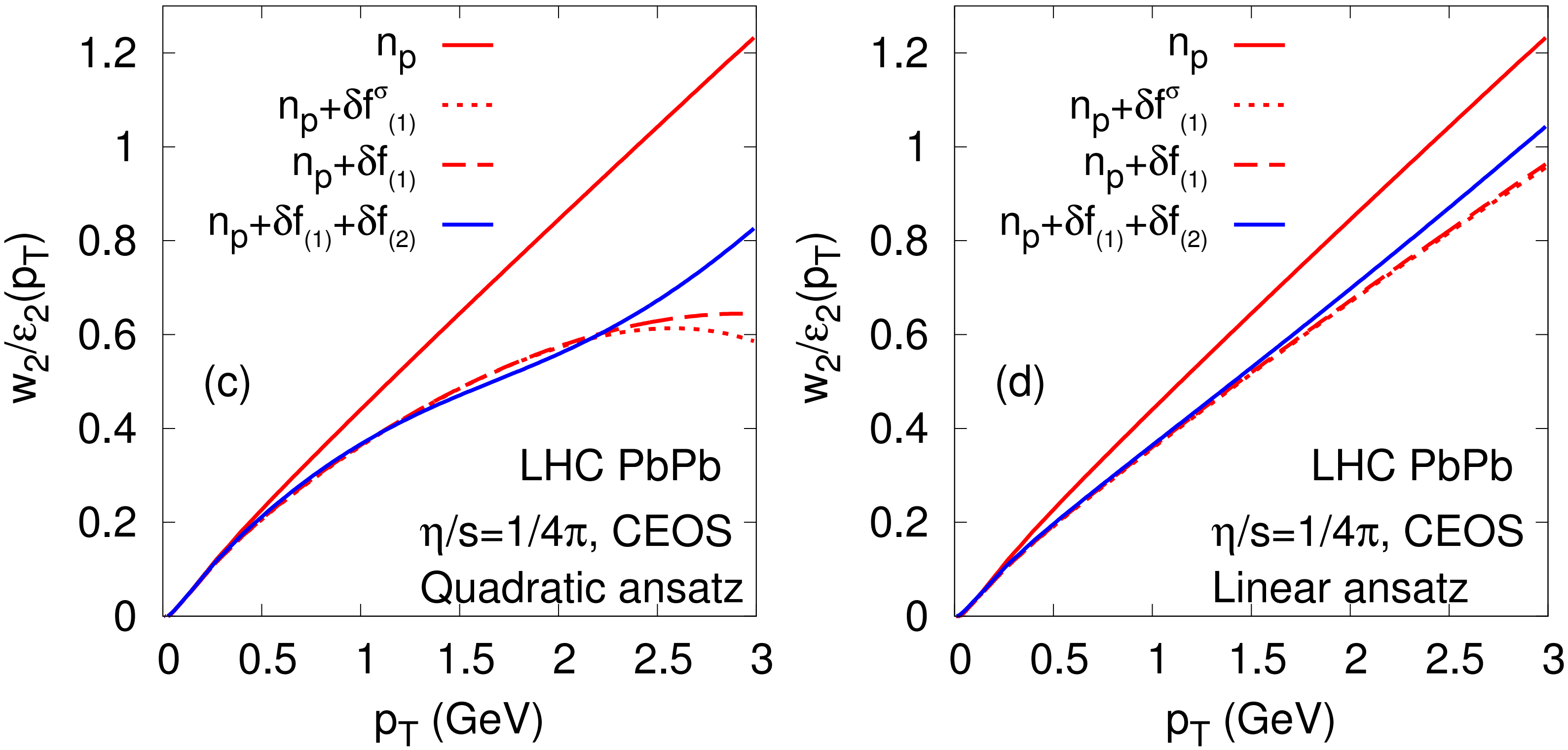
RHIC AuAu, with conformal EOS, $\alpha = 0$ and $\alpha = 1$:



- Difference between '$f^{\sigma}_{(1)}$' and '$f_{(1)}$' indicates the magnitude of gradients.
- α affects p_T dependence and the value of transport coefficients.

$\delta f_{(2)}$ and harmonic flow v_n

LHC PbPb, with conformal EOS, $\alpha = 0$ and $\alpha = 1$:



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$\delta f_{(2)}$ and harmonic flow v_n

What about non-conformal medium, with Lattice EOS?

1. Non-conformal terms are suppressed by $(\frac{1}{3} - c_s^2)$ and high p_T, for instance,

$$\delta f^{\sigma}_{(1)-\mathrm{non-conf}}(\mathbf{p}) \sim n'_p \left[\frac{p^\mu p^\nu \sigma_{\mu\nu}}{2T^3} + \left(-\frac{E_{\mathbf{p}}^2 - |\mathbf{p}|^2}{3T^3} + \frac{(\frac{1}{3} - c_s^2) E_{\mathbf{p}}^2}{T^3} \right) \nabla_\mu u^\mu \right]$$

2. For different particle species a, approximately

$$\eta/s = \frac{\sum_a \eta_a}{\sum_a s_a} = \eta_a / s_a$$

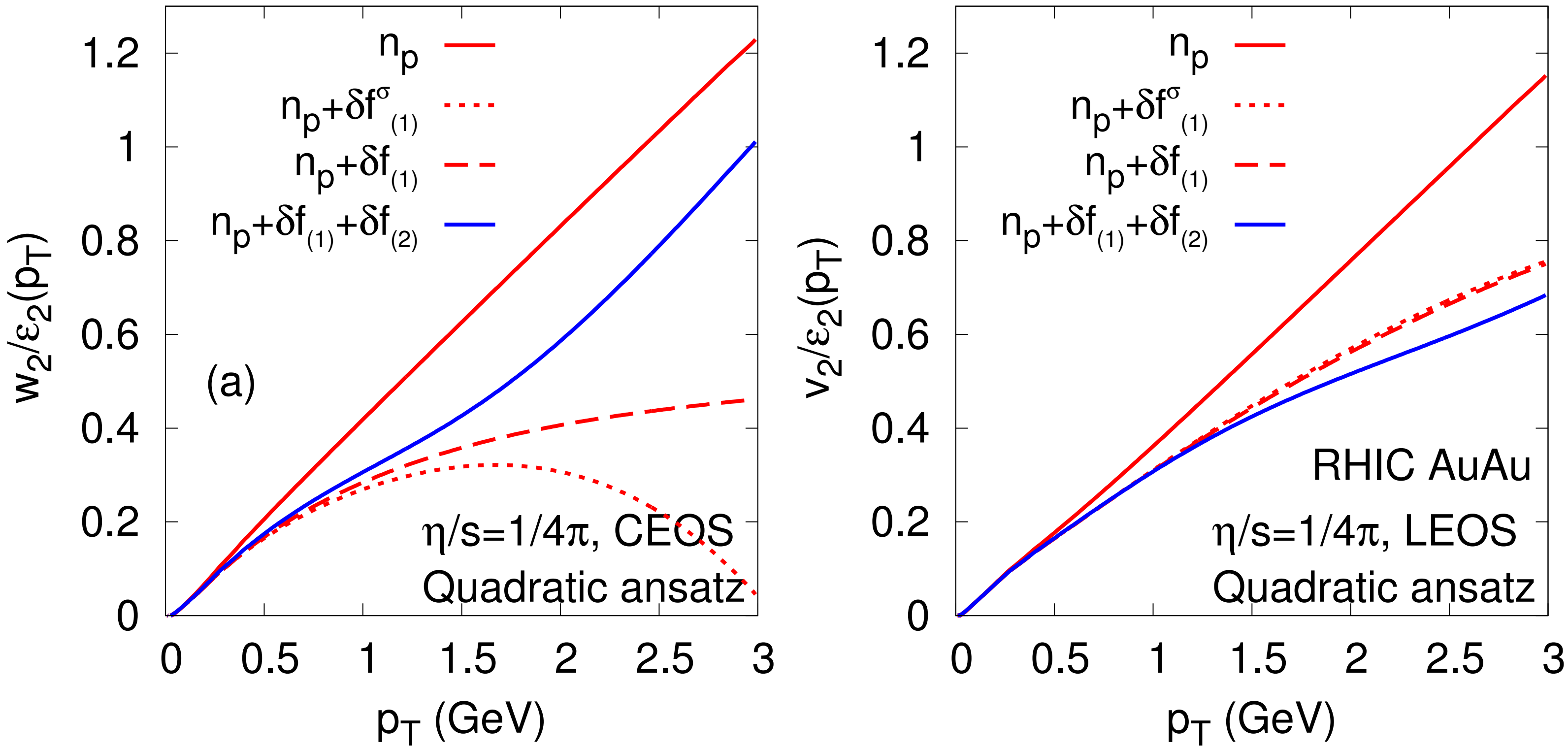
and

$$T^{\mu\nu} = \sum_a \int_p p^\mu p^\nu f$$

so for a multi-component gas the derivations are still valid.

$\delta f_{(2)}$ and harmonic flow v_n

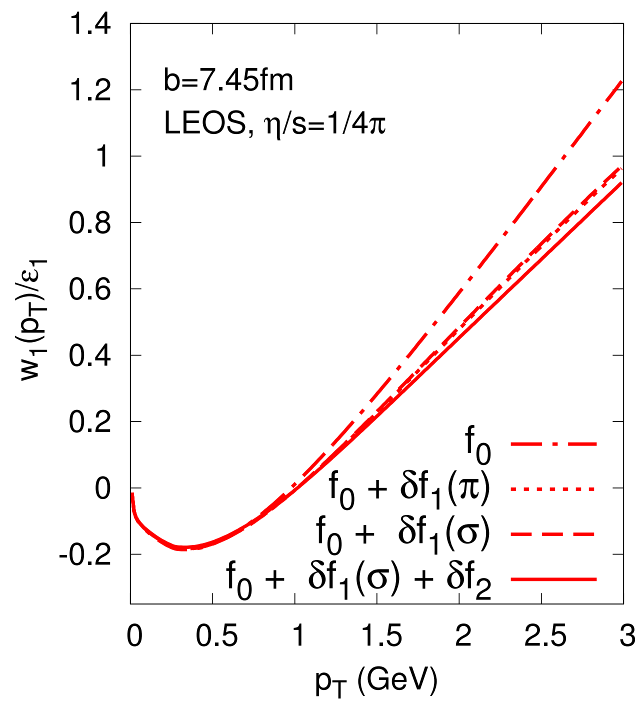
RHIC AuAu, for $\alpha = 0$ with conformal EOS and Lattice EOS:



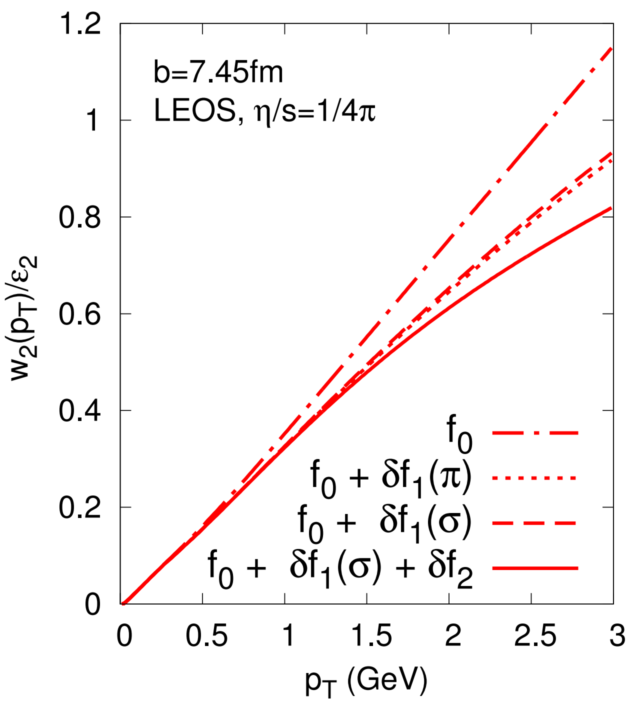
$\delta f_{(2)}$ and harmonic flow v_n

LHC PbPb:

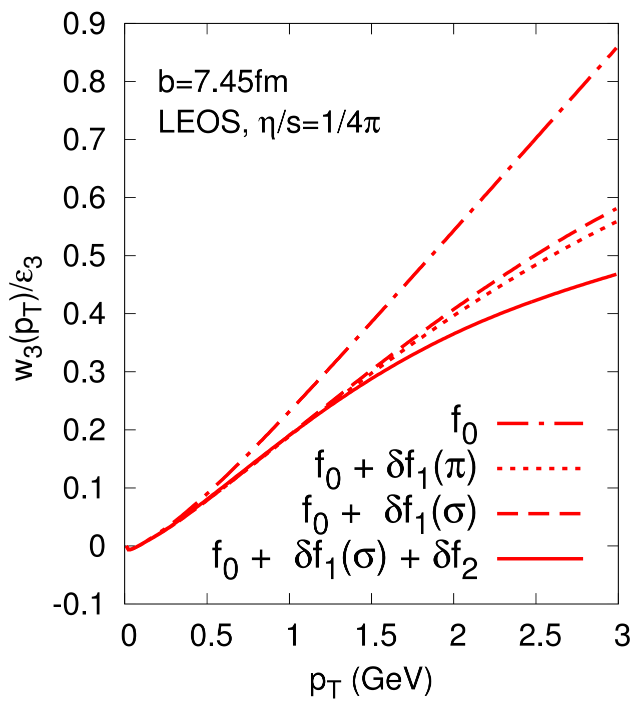
v_1



v_2

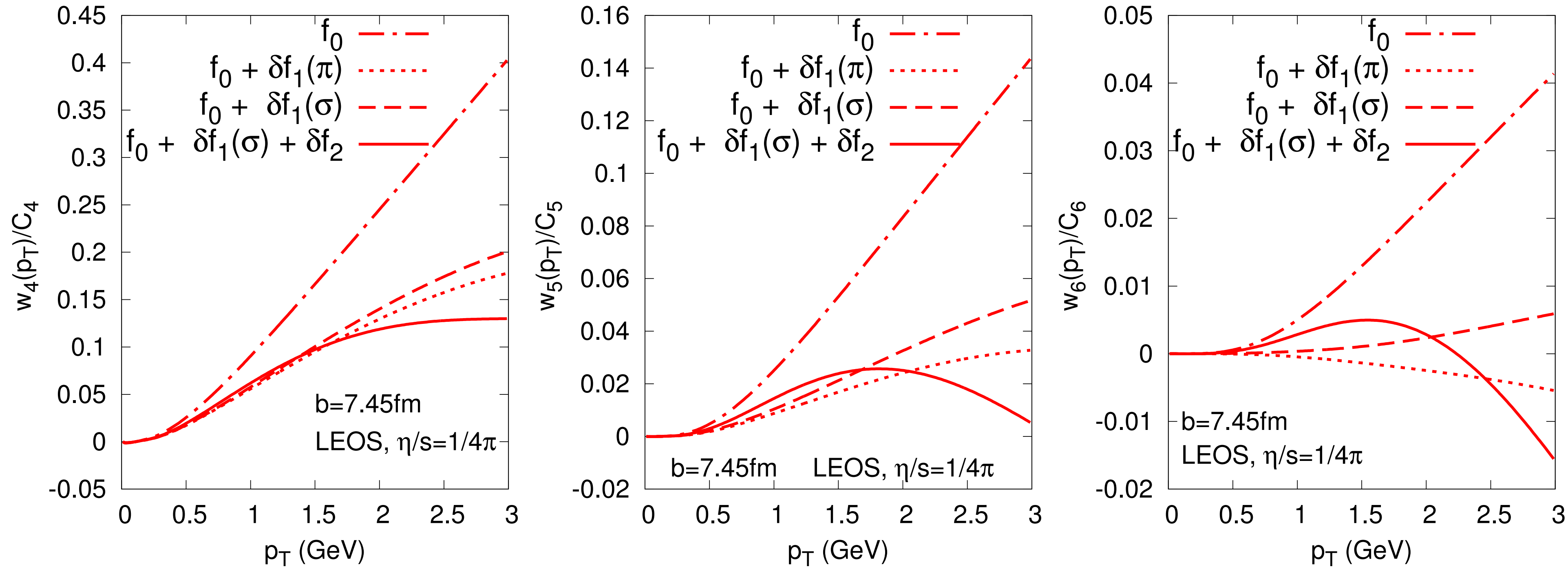


v_3



LHC PbPb:

v_4 v_5 v_6



Summary and conclusions

We have derived a consistent form of δf for the 2nd order viscous hydro:

- General procedure from solving Boltzmann equation order by order.
- 2nd order transport coefficients are fixed – kinetic approach.
- Formulation can be extended to other EOM's, such as Israel-Stewart hydro.
- δf affects harmonic spectrum, especially for large n and p_T.

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Thank you.

Back-up slides

Solving 2nd order viscous EOM – BRSSS

Equation of motion from the form of stress tensor, π as a dynamic variable

$$\pi^{\mu\nu} = -\eta\sigma^{\mu\nu} - \tau_\pi \left[{}^{\langle} D\pi^{\mu\nu\rangle} + \frac{4}{3}\pi^{\mu\nu}\nabla\cdot u \right] - \frac{\lambda_1}{\eta}\pi_\lambda^{\langle\,\mu}\pi^{\nu\lambda\rangle} - \frac{\lambda_2}{\eta}\pi_\lambda^{\langle\,\mu}\Omega^{\nu\lambda\rangle},$$