New paradigm in anisotropic flow analyses with multiparticle correlations

Ante Bilandzic
Technical University of Munich
Heavy Ion Meeting, Orsay, 06/10/2021

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Outline

- Introduction
  - Anisotropic flow
  - Multiparticle correlations and cumulants
- Focus of today's talk:
  - *'Higher order Symmetric Cumulants'*

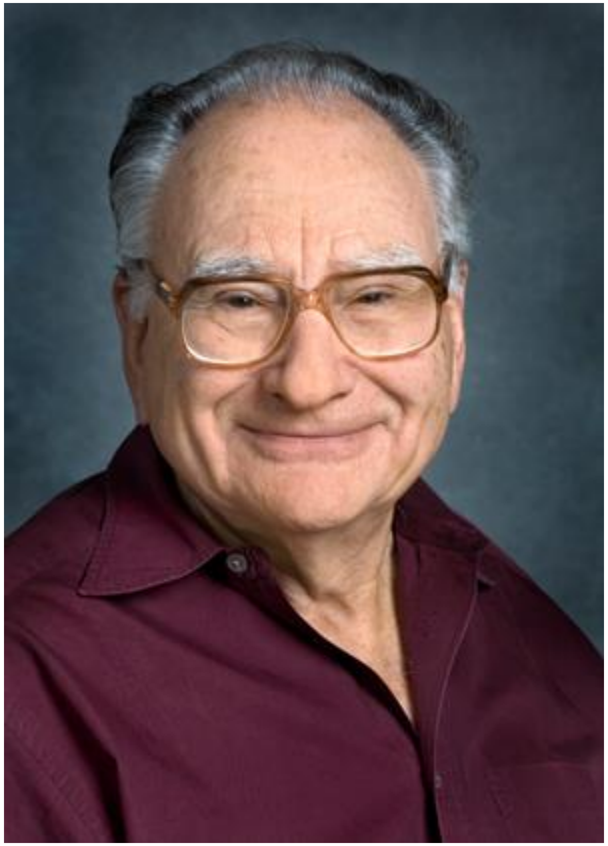
    C. Mordasini, AB, D. Karakoç, F. Taghavi, Phys. Rev. C **102**, 024907 (2020)
  - *'Multiharmonic Correlations of Different Flow Amplitudes…'*

    ALICE Collaboration, Phys. Rev. Lett. 127 (2021) 9, 092302
  - *'Multivariate cumulants in flow analyses: The Next Generation'*

    AB, M. Lesch, C. Mordasini, F. Taghavi, arXiv:2101.05619
  - *'Event-by-event cumulants of azimuthal angles'*

    A. Bilandzic, arXiv:2106.05760

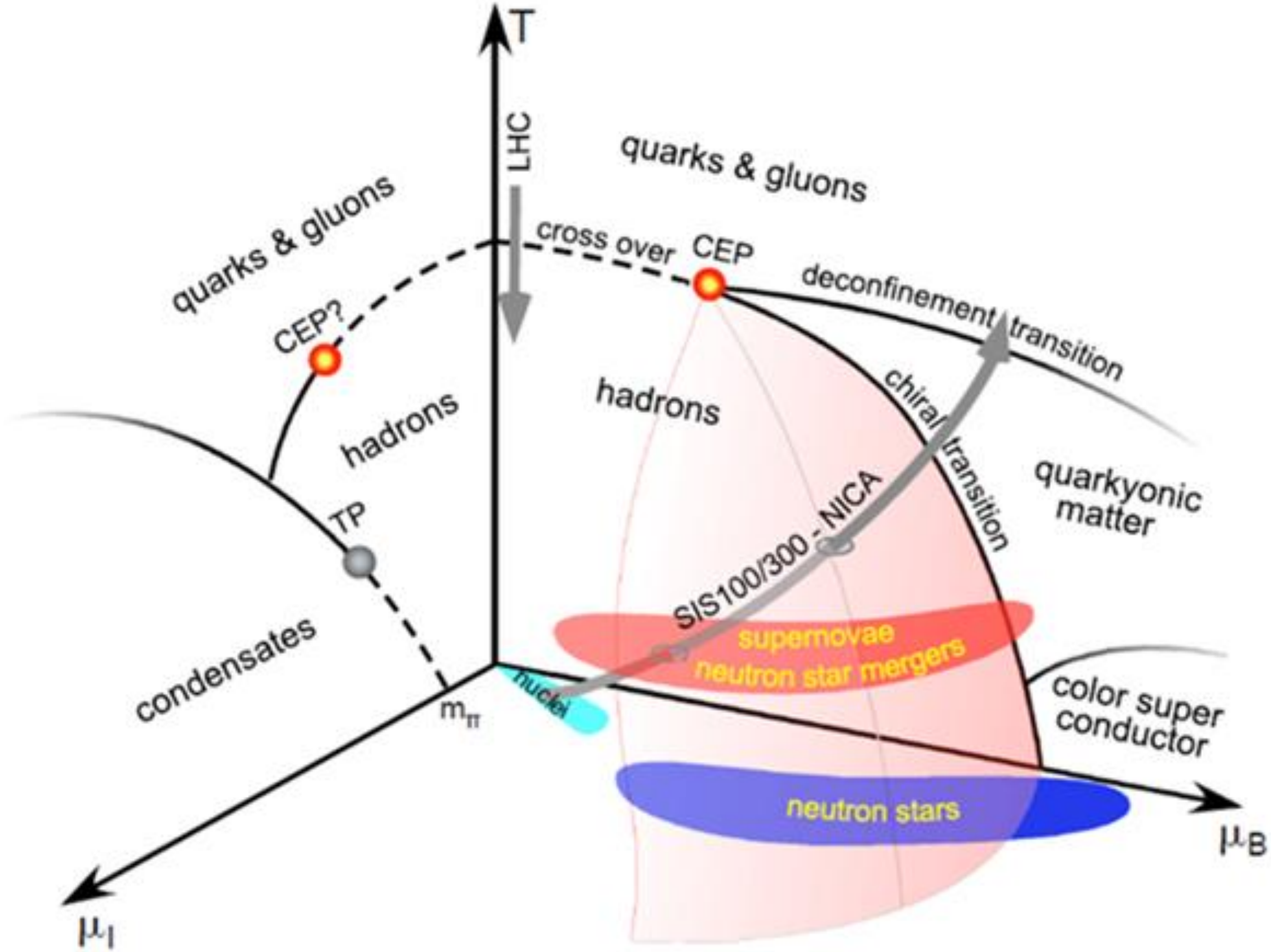
Tribute to Art Poskanzer (1931-2021)



Group photo from the workshop
'Initial State Fluctuations and Final State Correlations',
held at ECT* in Trento in July, 2012

A new state of matter: Quark-Gluon Plasma

- Phase diagram of Quantum Chromodynamics:



Which properties of QGP are we testing?

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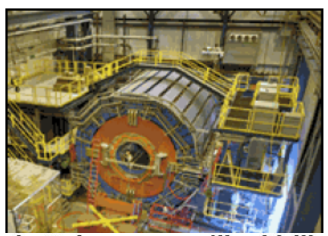
- Most notably:
  - Equation of state
  - Shear viscosity
  - Bulk viscosity
  - …

New State of Matter Is 'Nearly Perfect' Liquid

Physicists working at Brookhaven National Laboratory announced today that they have created what appears to be a new state of matter out of the building blocks of atomic nuclei, quarks and gluons. The researchers unveiled their findings--which could provide new insight into the composition of the universe just moments after the big bang--today in Florida at a meeting of the American Physical Society.

SCIENTIFIC AMERICAN

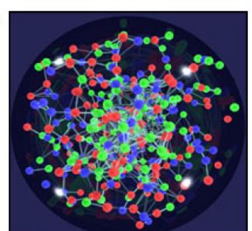


There are four collaborations, dubbed BRAHMS, PHENIX, PHOBOS and STAR, working at Brookhaven's Relativistic Heavy Ion Collider (RHIC). All of them study what happens when two interacting beams of gold ions smash into one another at great velocities, resulting in thousands
the researchers analyzed the patterns of the atom
found that the particles produced in the collisions t
school of fish does. Brookhaven's associate labor
physics, Sam Aronson, remarks that "the degree c
and extremely low viscosity of the matter being for
perfect liquid ever observed."

Early Universe was 'liquid-like'

Physicists say they have created a new state of hot, dense matter by crashing together the nuclei of gold atoms.

BBC NEWS



The impression is of matter that is more strongly interacting than predicted

The high-energy collisions prised open the nuclei to reveal their most basic particles, known as quarks and gluons.

The researchers, at the US Brookhaven National Laboratory, say these particles were seen to behave as an almost perfect "liquid".

The work is expected to help scientists explain the conditions that existed just milliseconds after the Big Bang.

Universe May Have Begun as Liquid, Not Gas

Associated Press
Tuesday, April 19, 2005; Page A05

The Washington Post

New results from a particle collider suggest that the universe behaved like a liquid in its earliest moments, not the fiery gas that was thought to have pervaded the first microseconds of existence.

Early Universe was a liquid

Quark-gluon blob surprises particle physicists.

by Mark Peplow
news@nature.com

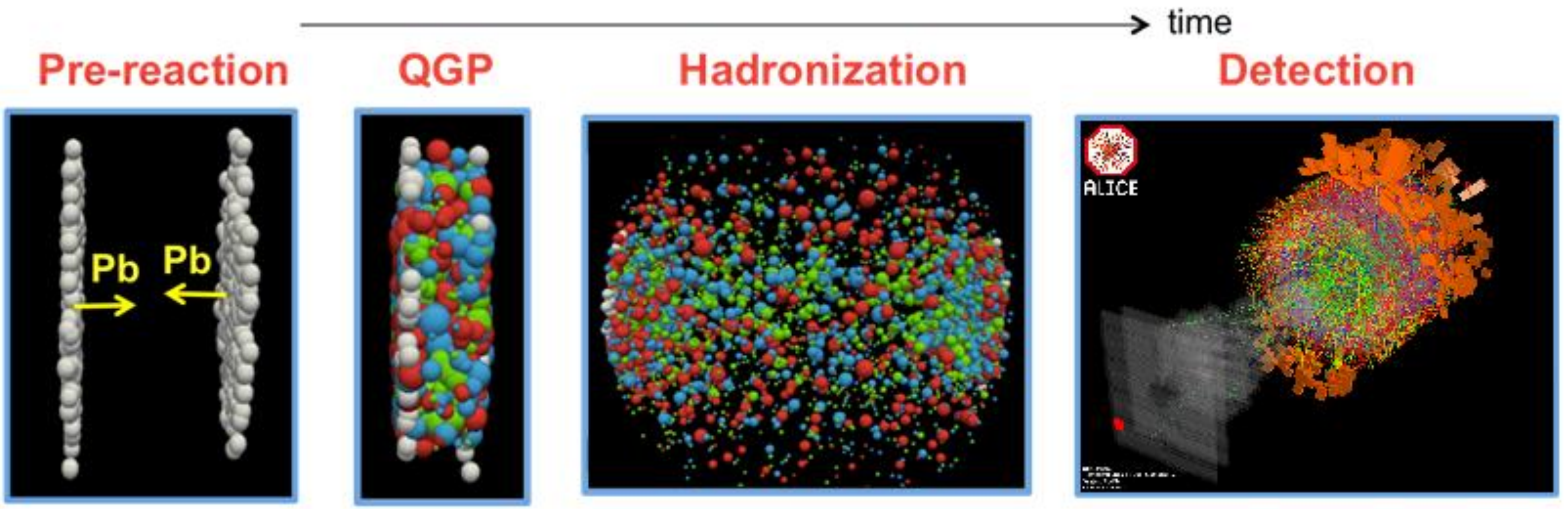
nature

The Universe consisted of a perfect liquid in its first moments, according to results from an atom-smashing experiment.

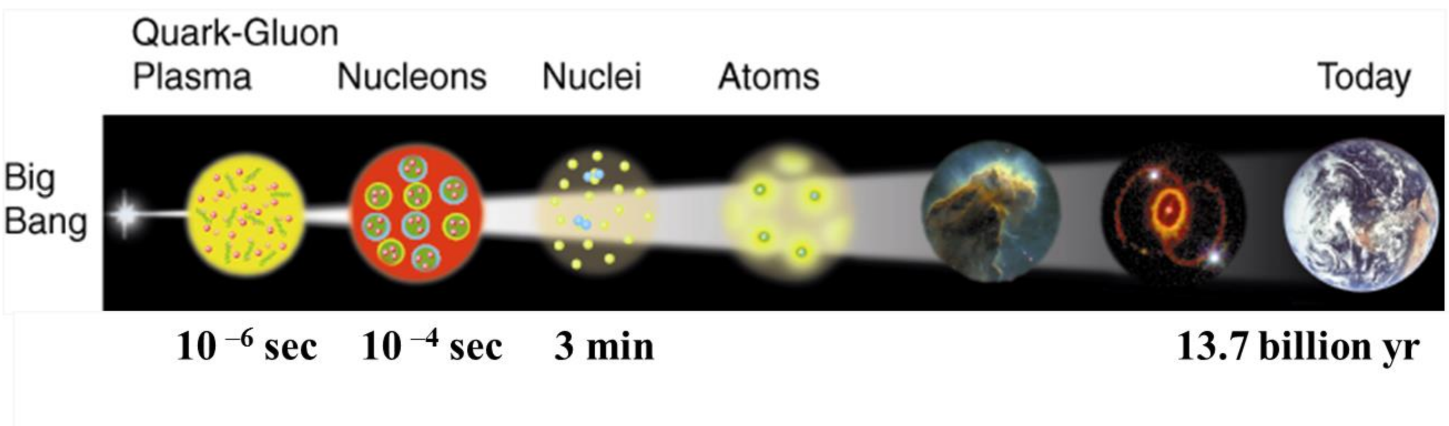
Scientists at the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory on Long Island, New York, have spent five years searching for the quark-gluon plasma that is thought to have filled our Universe in the first microseconds of its existence. Most of them are now convinced they have found it. But, strangely, it seems to be a liquid rather than the expected hot gas.

How can we produce Quark-Gluon Plasma?

- Heavy-ion collisions at collider experiments at LHC:

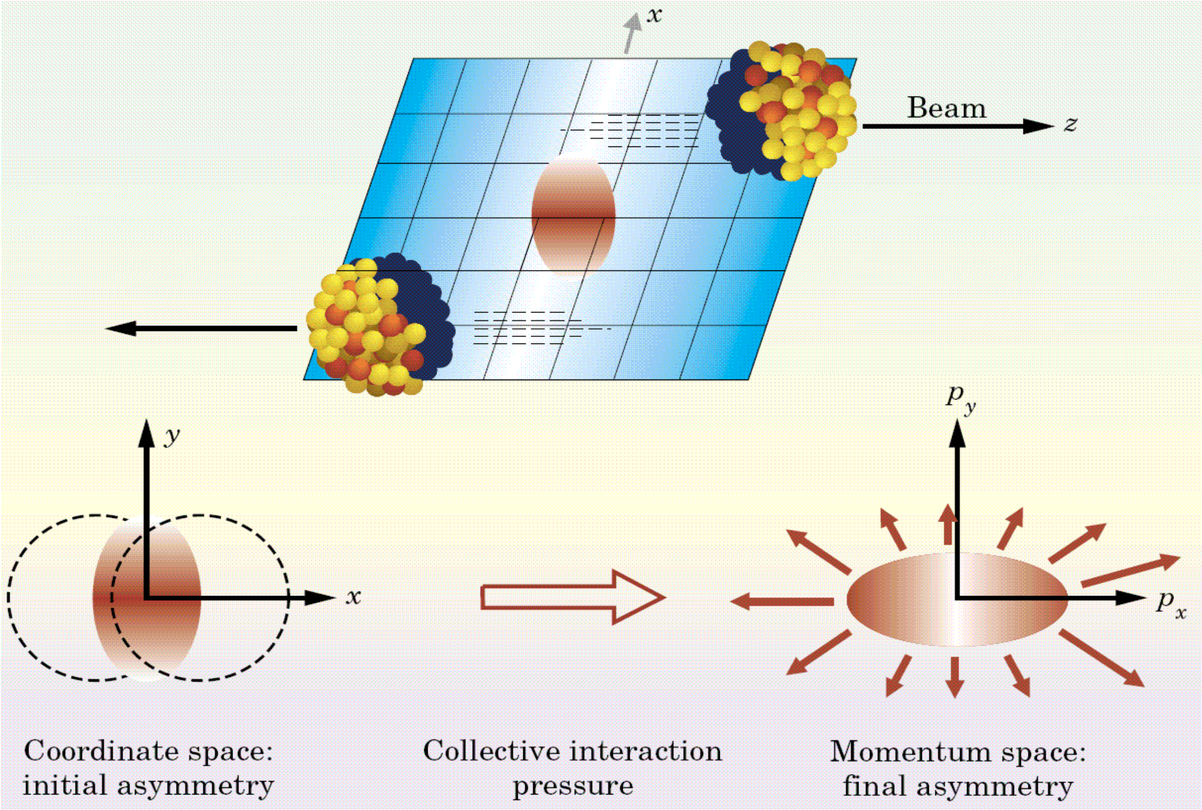


- QGP filled the Universe few microseconds after Big Bang:



Anisotropic flow phenomenon

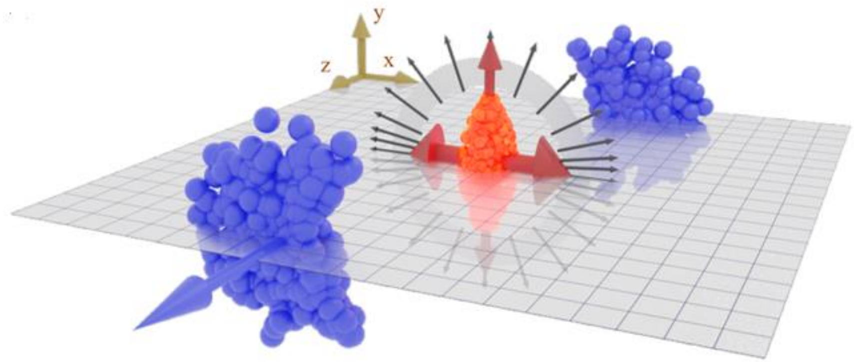
- Transfer of anisotropy from the initial coordinate space into the final momentum space via the thermalized medium:



J.-Y. Ollitrault, Phys. Rev. D **46**, 229 (1992)

Two pillars of flow development

- Anisotropic flow will develop in heavy-ion collisions only if both of the following two requirements are met:
  - Initial anisotropic volume in coordinate space: **'Trigger'**
  - Thermalized medium: **'Transfer'**

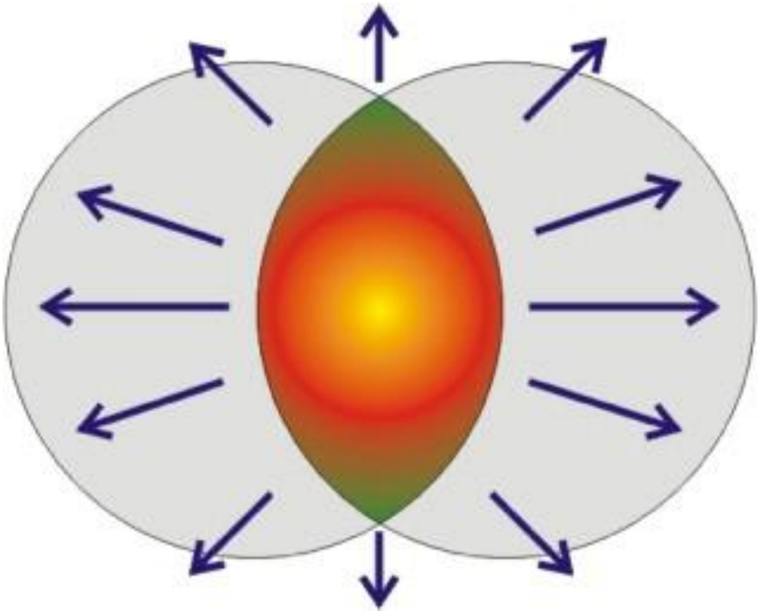


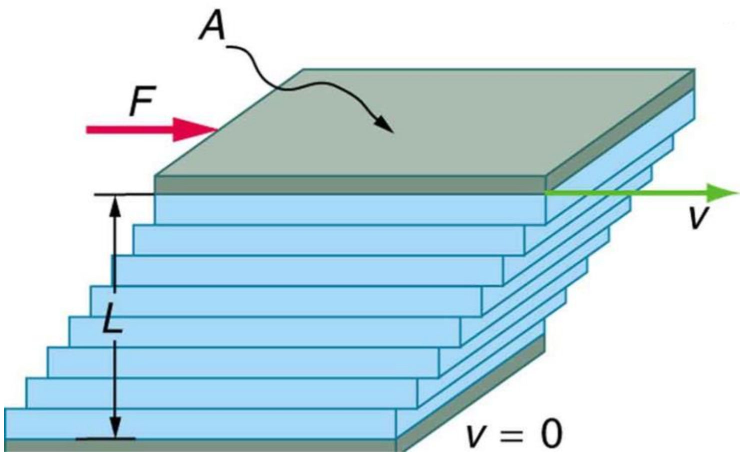
Credits: D.D. Chinellato, ICHEP 2020

- Anisotropic flow is a sensitive probe both of **initial conditions** in heavy-ion collisions, and of QGP's **transport properties** (e.g. of its shear viscosity)

Hydrodynamic flow in-plane

- Non-trivial effect which is sensitive to transport coefficients of QGP (e.g. its shear viscosity)





If anisotropic flow has developed, neighboring layers are moving at different relative velocities, parallel displacement is opposed by QGP's shear viscosity

large anisotropic flow ⇔ small shear viscosity

Fourier series

- We use Fourier series to describe anisotropic emission of particles in the plane transverse to the beam direction after every heavy-ion collision:

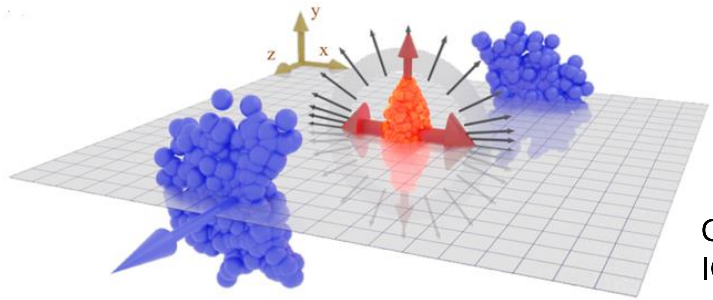
$$f(\varphi) = \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{\infty} v_n \cos[n(\varphi - \Psi_n)]\right]$$

- v_n : flow amplitudes
- Ψ_n : symmetry planes
- Anisotropic flow is quantified with v_n and Ψ_n
  - v_1 is directed flow
  - v_2 is elliptic flow
  - v_3 is triangular flow
  - v_4 is quadrangular flow, etc.

S. Voloshin and Y. Zhang, Z.Phys. C70 (1996) 665-672

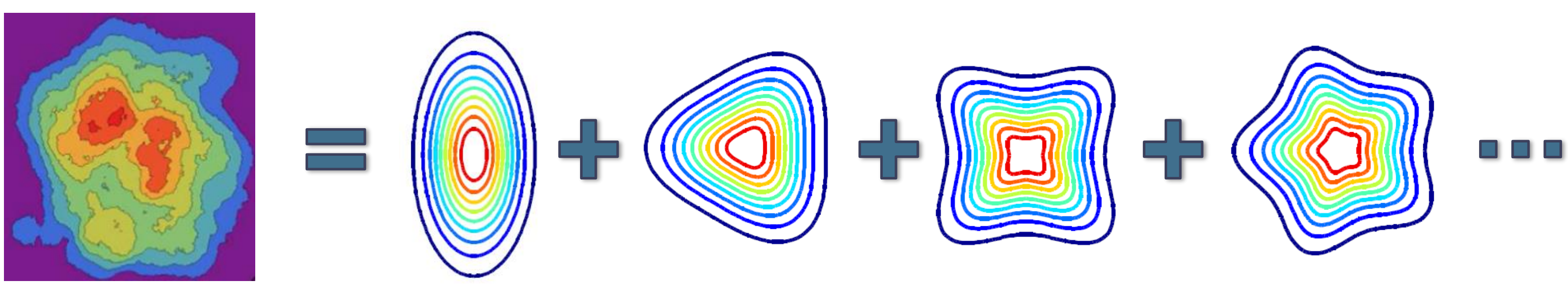
Fourier series

- In non-central heavy-ion collisions, due to collision geometry the initial volume is almond shaped (ellipsoidal)
  - Dominant harmonic is v_2 (elliptic flow)

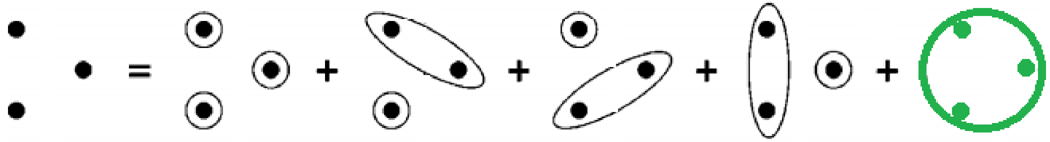


Credits: D.D. Chinellato, ICHEP 2020

- In most central (head-on) collisions, due to fluctuations of participating nucleons any shape can develop, all (lower) order harmonics are equally probable



Multiparticle correlations and cumulants



Multiparticle azimuthal correlations

- The most general result, which relates multiparticle azimuthal correlators and flow degrees of freedom:

$$\langle\cos[n_1\varphi_1+\cdots+n_k\varphi_k)]\rangle = v_{n_1}\cdots v_{n_k}\cos[n_1\Psi_{n_1}+\cdots+n_k\Psi_{n_k})]$$

R. S. Bhalerao, M. Luzum and J.-Y. Ollitrault, Phys. Rev. C **84** 034910 (2011)

- Flow amplitudes v_n and symmetry planes Ψ_n
- A lot of non-trivial and independent flow observables for different choices of harmonics $\boldsymbol{n_i}$
  - Examples: 2- and 4-particle azimuthal correlations

$$\langle\cos[n(\varphi_1-\varphi_2)]\rangle = v_n^2$$
$$\langle\cos[n(\varphi_1+\varphi_2-\varphi_3-\varphi_4)]\rangle = v_n^4$$

Scaling of stat. and sys. errors

- Scaling of statistical uncertainty (*N* is number of events, *M* is multiplicity, *v* is flow strength, *k* is order of correlator):

$$\sigma_v \sim \frac{1}{\sqrt{N}} \frac{1}{M^{k/2}} \frac{1}{v^{k-1}}$$

- Scaling of non-collective contribution:

$$\delta_k \sim \frac{1}{M^{k-1}}$$

- For both reasons, multiparticle correlations are precision technique only for: a) large multiplicities, b) large flow

2-particle cumulants in general

- X_i denotes the general i-th stochastic variable
- The most general decomposition of 2-particle correlation is:

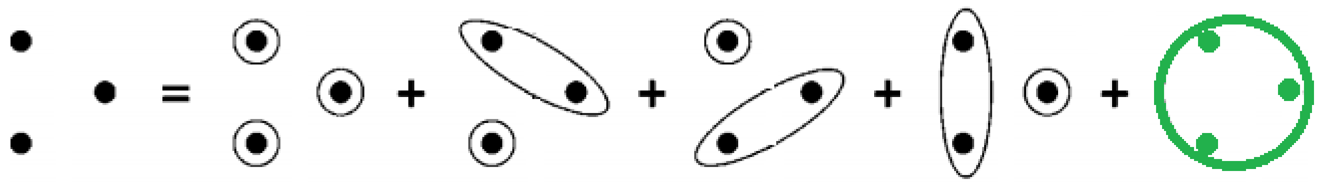
$$\langle X_1 X_2 \rangle = \langle X_1 \rangle \langle X_2 \rangle + \langle X_1 X_2 \rangle_c$$

- By definition, the 2nd term on RHS is 2-particle cumulant
- Cumulants cannot be measured directly, however:

$$\langle X_1 X_2 \rangle_c = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$$

3-particle cumulants in general

- The most general decomposition of 3-particle correlation is:



- Or written mathematically:

$$\begin{aligned}\langle X_1 X_2 X_3 \rangle &= \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle \\ &+ \langle X_1 X_2 \rangle_c \langle X_3 \rangle + \langle X_1 X_3 \rangle_c \langle X_2 \rangle + \langle X_2 X_3 \rangle_c \langle X_1 \rangle \\ &+ \langle X_1 X_2 X_3 \rangle_c\end{aligned}$$

- The key point: 2-particle cumulants were expressed independently in terms of measured correlations in previous step!

$$\langle X_1 X_2 \rangle_c = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$$

3-particle cumulants in general

- Working recursively from higher to lower orders, we eventually have 3-particle cumulant expressed in terms of measured 3-, 2-, and 1-particle averages

$$\begin{aligned}\langle X_1 X_2 X_3 \rangle_c &= \langle X_1 X_2 X_3 \rangle \\ &- \langle X_1 X_2 \rangle \langle X_3 \rangle - \langle X_1 X_3 \rangle \langle X_2 \rangle - \langle X_2 X_3 \rangle \langle X_1 \rangle \\ &+ 2 \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle\end{aligned}$$

- In the same way, cumulants can be expressed in terms of measurable averages for any number of particles
  - The number of terms grows rapidly

Cumulants in flow analyses

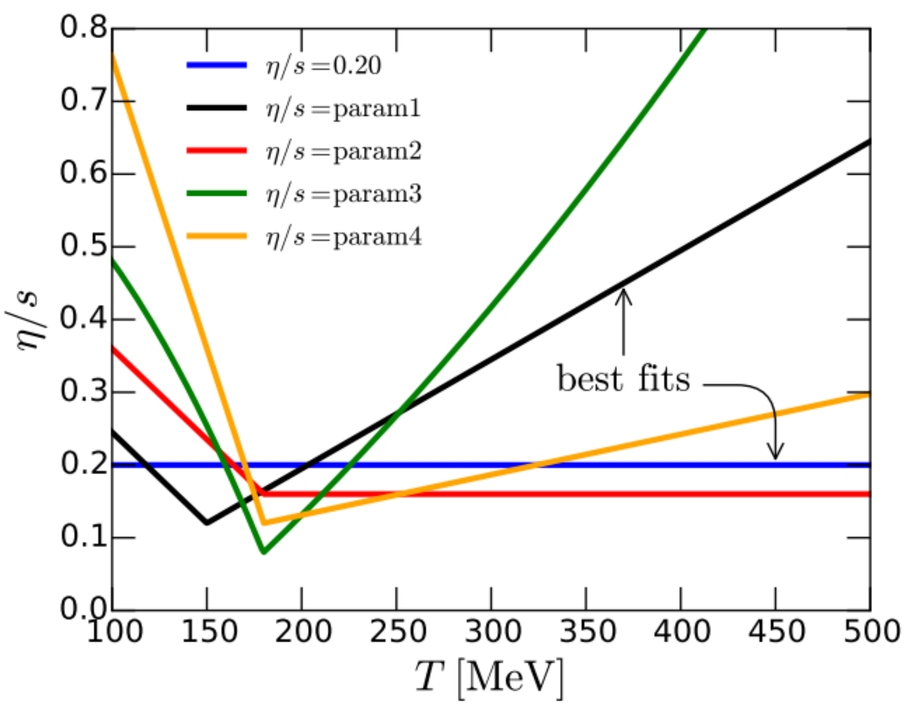
- Cumulants were introduced in flow analyses by Ollitrault *et al* in two seminal papers:

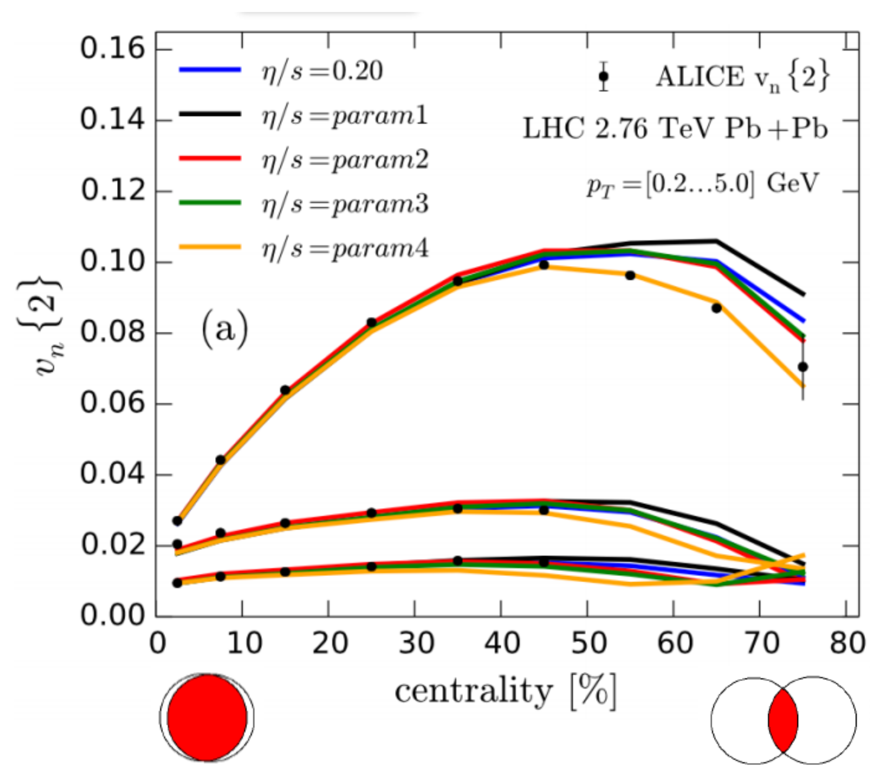
  N. Borghini, P. M. Dinh, and J.-Y. Ollitrault, Phys. Rev. C **63**, 054906 (2001)
  N. Borghini, P. M. Dinh, and J.-Y. Ollitrault, Phys. Rev. C **64**, 054901 (2001)

- Traditionally, azimuthal angles are chosen as fundamental observables in the cumulant expansion
- Based on this approach, one derives e.g. $v_n\{4\}$ observable
  - It gives an estimate for flow harmonic v_n by using 4-particle azimuthal cumulant (not 4-p azimuthal correlator!)
  - For large multiplicities, $v_n\{4\}$ suppresses well nonflow effects
- But this traditional approach ('old paradigm') yields **very weird and inconsistent results** when applied to the correlations of different flow amplitudes

‘Classical’ flow observables

- Insensitivity to temperature dependence of η/s





H. Niemi, K. J. Eskola, R. Paatelainen, Phys. Rev. C 93, 024907 (2016)

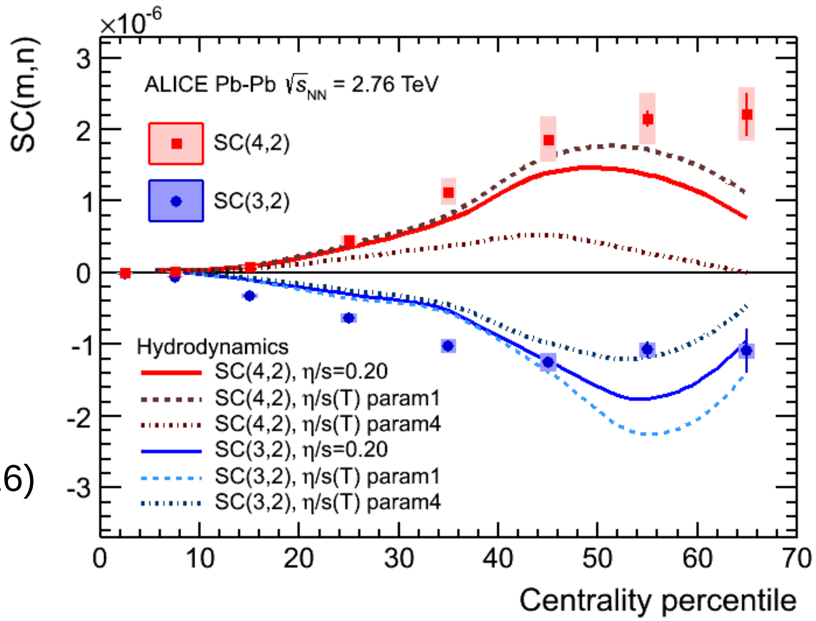
Symmetric Cumulants SC(*m*,*n*)

- How to quantify experimentally the correlation between two different flow amplitudes?
  - Symmetric Cumulants (Section IVC in Phys. Rev. C **89** (2014) no.6, 064904)

$$\langle\langle\cos(m\varphi_1+n\varphi_2-m\varphi_3-n\varphi_4)\rangle\rangle_c = \langle\langle\cos(m\varphi_1+n\varphi_2-m\varphi_3-n\varphi_4)\rangle\rangle - \langle\langle\cos[m(\varphi_1-\varphi_2)]\rangle\rangle\,\langle\langle\cos[n(\varphi_1-\varphi_2)]\rangle\rangle = \langle v_m^2 v_n^2\rangle - \langle v_m^2\rangle\langle v_n^2\rangle$$

- SC observables are sensitive to differential η/s(T) parametrizations
- Individual flow amplitudes are dominated by averages ⟨η/s(T)⟩
- Independent constraints both on initial conditions and QGP properties

ALICE Collaboration, Phys. Rev. Lett. 117, 182301 (2016)



Choice of fundamental observable

- Cumulants as used in flow analyses in the last ~20 years:
  1. Cumulant expansion is performed on azimuthal angles
  2. Azimuthal correlators which are not isotropic are dropped
  3. The final result is merely re-expressed in terms of flow degrees of freedom v_n and Ψ_n via the analytic relation

$$\langle\cos[n_1\varphi_1+\cdots+n_k\varphi_k)]\rangle = v_{n_1}\cdots v_{n_k}\cos[n_1\Psi_{n_1}+\cdots+n_k\Psi_{n_k})]$$

R. S. Bhalerao, M. Luzum and J.-Y. Ollitrault, Phys. Rev. C **84** 034910 (2011)

- Few additional remarks:
  - Cumulants of v_n and v_n^2 are in general different
  - v_n and Ψ_n have different properties (e.g. with respect to Lorentz invariance)

The root of the problem

- General 2-particle cumulant

$$\langle X_1 X_2 \rangle_c = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$$

  - Old paradigm: fundamental observable is an angle

$$X_1 \to e^{in\varphi_1}\,, \quad X_2 \to e^{-in\varphi_2}$$

  - New paradigm: fundamental observable is a flow amplitude

$$X_1 \to v_n^2\,, \quad X_2 \to v_m^2$$

- Both choices yielded **accidentally the same results** for SC(m,n) observables
- But results for SC(k,l,m), SC(k,l,m,n), etc., are in general different
  - Which paradigm is correct in general?

C. Mordasini, AB, D. Karakoç, F. Taghavi: *'Higher order Symmetric Cumulants'*, Phys. Rev. C **102**, 024907 (2020)

Generalization: $SC(k,l,m)$, $SC(k,l,m,n)$, ...

- New paradigm:

  1/ Cumulant expansion directly on flow amplitudes v^2:

$$\mathrm{SC}(k,l,m) \equiv \left\langle v_k^2 v_l^2 v_m^2 \right\rangle - \left\langle v_k^2 v_l^2 \right\rangle \left\langle v_m^2 \right\rangle - \left\langle v_k^2 v_m^2 \right\rangle \left\langle v_l^2 \right\rangle - \left\langle v_l^2 v_m^2 \right\rangle \left\langle v_k^2 \right\rangle + 2 \left\langle v_k^2 \right\rangle \left\langle v_l^2 \right\rangle \left\langle v_m^2 \right\rangle$$

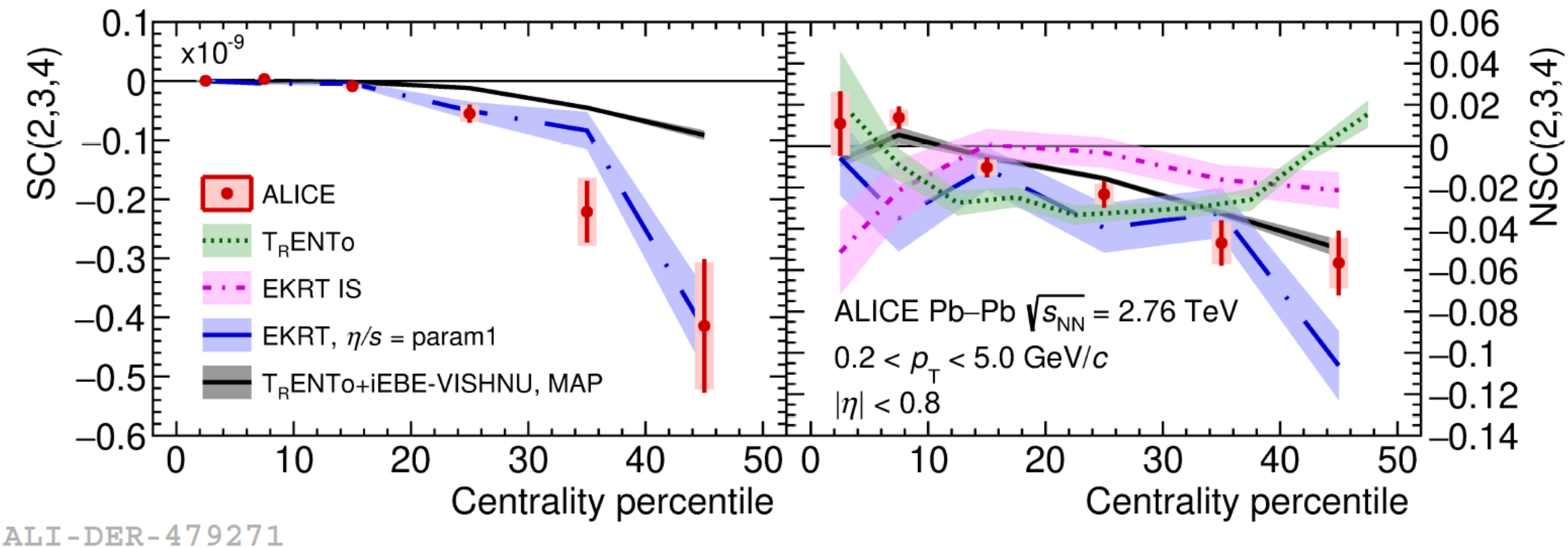
  2/ Azimuthal angles are used merely to build an estimator for the above observable:

$$\begin{aligned}
\mathrm{SC}(k,l,m) = &\langle\langle\cos[k\varphi_1+l\varphi_2+m\varphi_3-k\varphi_4-l\varphi_5-m\varphi_6]\rangle\rangle \\
&- \langle\langle\cos[k\varphi_1+l\varphi_2-k\varphi_3-l\varphi_4]\rangle\rangle \langle\langle\cos[m(\varphi_5-\varphi_6)]\rangle\rangle \\
&- \langle\langle\cos[k\varphi_1+m\varphi_2-k\varphi_5-m\varphi_6]\rangle\rangle \langle\langle\cos[l(\varphi_3-\varphi_4)]\rangle\rangle \\
&- \langle\langle\cos[l\varphi_3+m\varphi_4-l\varphi_5-m\varphi_6]\rangle\rangle \langle\langle\cos[k(\varphi_1-\varphi_2)]\rangle\rangle \\
&+ 2 \langle\langle\cos[k(\varphi_1-\varphi_2)]\rangle\rangle \langle\langle\cos[l(\varphi_3-\varphi_4)]\rangle\rangle \langle\langle\cos[m(\varphi_5-\varphi_6)]\rangle\rangle
\end{aligned}$$

C. Mordasini, AB, D. Karakoç, F. Taghavi: *'Higher order Symmetric Cumulants'*, Phys. Rev. C **102**, 024907 (2020)

$SC(k,l,m)$ in ALICE

- C. Mordasini, Ph.D. thesis, *"Generalisation of the Cumulants of … "*

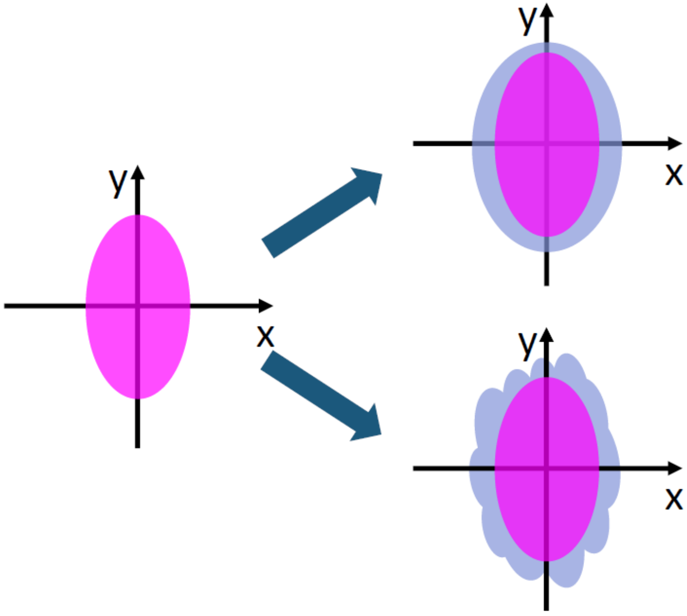


Comparison with the state-of-the-art models: Development of genuine multiharmonic correlations during hydrodynamic evolution

ALICE Collaboration, Phys. Rev. Lett. 127 (2021) 9, 092302

Shear vs. bulk viscosities

- Can we separate the effects of shear (η) and bulk (ξ) viscosities?



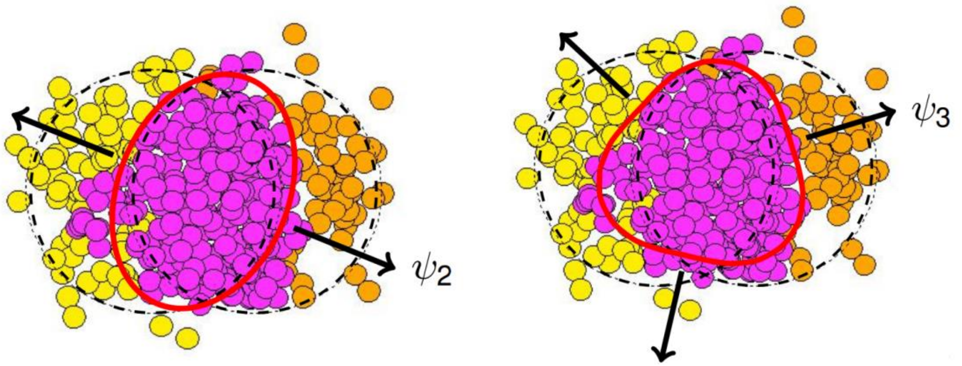
- Isotropic fluctuations
  - Neighbouring layers move at equal velocities
  - Generally preserves the ellipse shape
  - Main sensitivity to ξ/s
- Shape fluctuations
  - Neighbouring layers move at different velocities
  - Sensitivity to η/s

Credits: C. Mordasini

- We need new observables which can separate these different sources of fluctuations

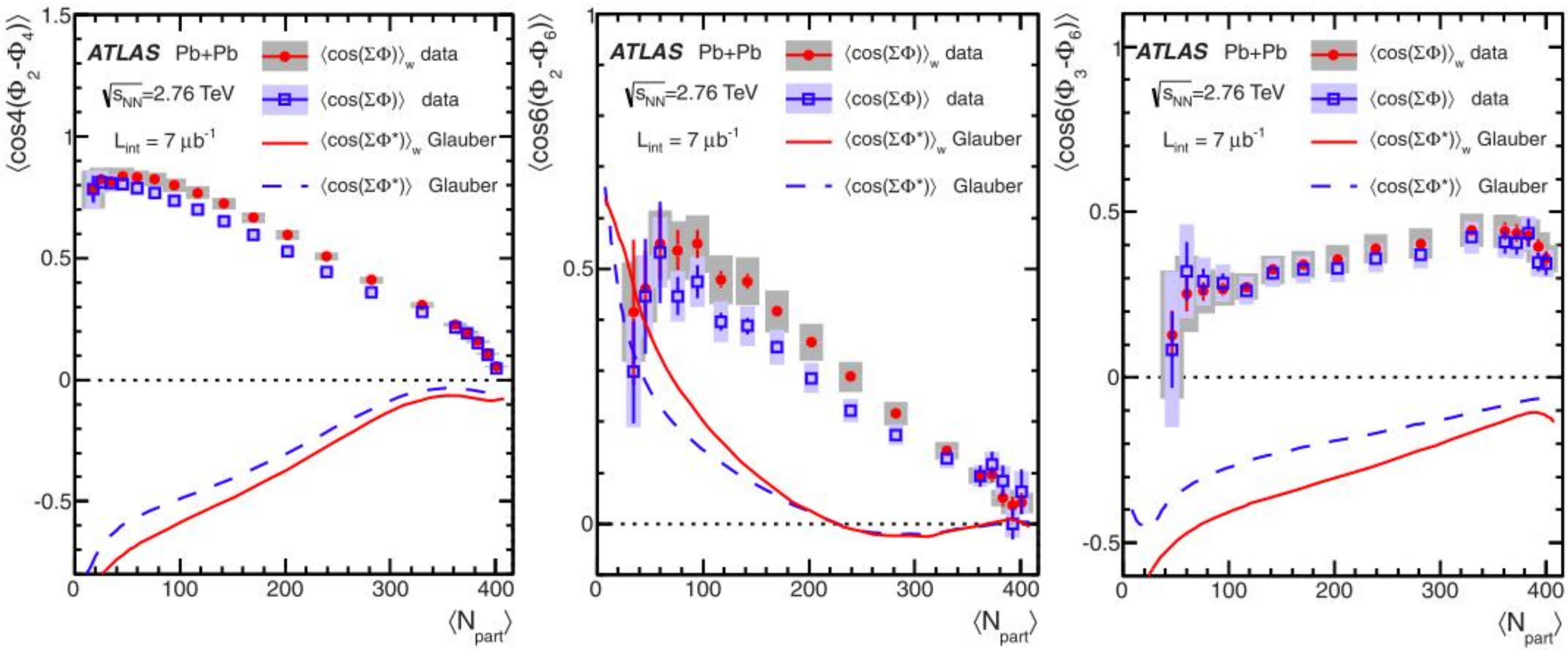
‘New estimator for symmetry plane correlations in anisotropic flow analyses’

A. Bilandzic, M. Lesch, F. Taghavi, Phys. Rev. C **102**, 024910 (2020)



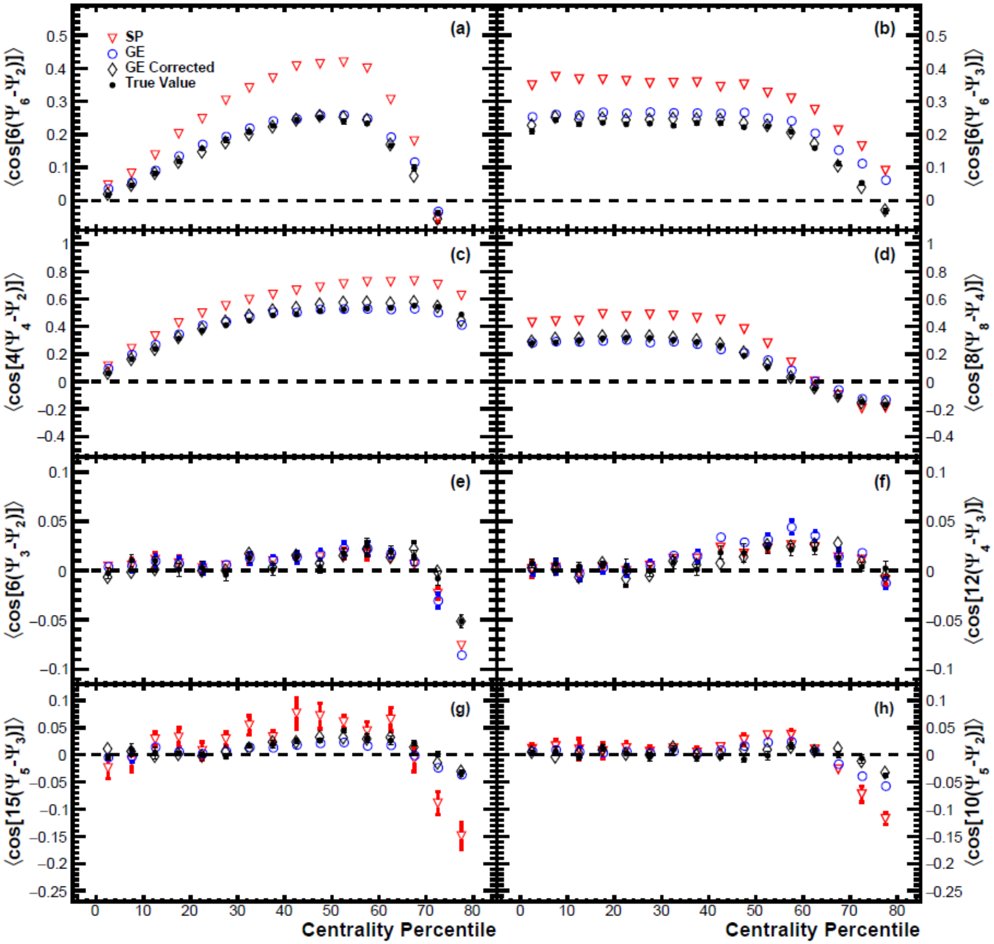
Symmetry plane correlations

- ATLAS: Phys. Rev. C **90**, 024905



- Correlations of symmetry planes in coordinate space are not equal to correlations of symmetry planes in momentum space

New estimator for symmetry plane correlations



- Clear improvement over other existing estimators (e.g. the one based on traditional Scalar Product (SP) method)
- For centralities in which SP estimator (red markers) fails to reproduce the true values (black markers), our new estimator is still doing a great job!
- First experimental results available…

AB, M. Lesch, F. Taghavi, Phys. Rev. C **102**, 024910 (2020)

‘Multivariate cumulants in flow analyses: The Next Generation’

A. Bilandzic, M. Lesch, C. Mordasini, F. Taghavi, https://arxiv.org/abs/2101.05619

Fundamental properties of cumulants

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- We reviewed everything from scratch and supported proofs for:
  - Statistical independence
  - Reduction
  - Semi-invariance
  - Homogeneity
  - Multilinearity
  - Additivity
  - …

For all technical details, see Section II and Appendix A in arXiv:2101.05619

- The main strategy in this technical paper is divided into two steps:
  - Confront all existing observables in the field named cumulants with these fundamental properties
  - For the ones which fail to satisfy them, provide the alternative definitions which do satisfy all fundamental properties of cumulants

AB, M. Lesch, C. Mordasini, F. Taghavi, arXiv:2101.05619

Main conclusions

- **The main conclusion #1:** One cannot perform cumulant expansion in one set of stochastic observables, then in the resulting expression perform the transformation to some new set of observables, and then claim that the cumulant properties are preserved in the new set of observables
  - After such transformation, the fundamental properties of cumulants are lost in general

- **The main conclusion #2:** The formal properties of cumulants are valid only if there are no underlying symmetries due to which some terms in the cumulant expansion would vanish identically
  - Due to symmetries, $\left\langle e^{in\varphi_i}\right\rangle = 0$, $\left\langle e^{in(\varphi_i+\varphi_j)}\right\rangle = 0$, etc., all vanish
  - There are no obvious symmetries for $\left\langle v_k^2\right\rangle$, $\left\langle v_k^2 v_l^2\right\rangle$, etc., to vanish

AB, M. Lesch, C. Mordasini, F. Taghavi, arXiv:2101.05619

Necessary conditions for cumulants

- From the fundamental properties of cumulants (statistical independence, reduction, semi-invariance, homogeneity, multilinearity, additivity, etc.), we have established the following two simple necessary conditions:

1. We take temporarily that in the definition of $\lambda(X_1, \ldots, X_N)$ all observables $X_1, \ldots, X_N$ are statistically independent and factorize all multivariate averages into the product of single averages $\Rightarrow$ the resulting expression must reduce identically to 0;

2. We set temporarily in the definition of $\lambda(X_1, \ldots, X_N)$ all observables $X_1, \ldots, X_N$ to be the same and equal to X $\Rightarrow$ for the resulting expression it must hold that

$$\lambda(aX + b) = a^N \lambda(X)\,, \tag{23}$$

where a and b are arbitrary constants, and N is the number of observables in the starting definition of $\lambda(X_1, \ldots, X_N)$.

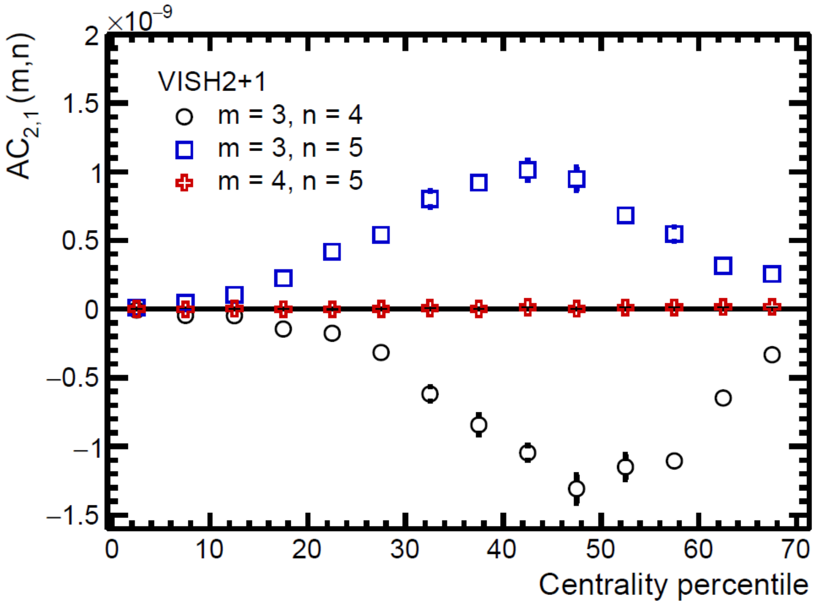
Multivariate observable is a multivariate cumulant only if it satisfies both above requirements (arXiv:2101.05619)

Reconciliation

- New flow observables ('The Next Generation') which do satisfy all formal mathematical properties of cumulants:
  - 'Symmetric and Asymmetric Cumulants' (genuine multiharmonic correlations of flow amplitudes)
    - See arXiv:1901.06968 and Sec. V in arXiv:2101.05619
  - 'Cumulants of symmetry plane correlations'
    - See Sec. VI in arXiv:2101.05619
  - 'Event-by-event cumulants of azimuthal angles'
    - See Sec. IV in arXiv:2101.05619 and arXiv: 2106.05760

Asymmetric Cumulants (AC)

- Generalization of Symmetric Cumulants
- Fundamental observable is v^2
  - Choice driven by experiment: The simplest flow moment which can be estimated experimentally with azimuthal correlations
- Each of these observables is insensitive to lower-order correlations, because they satisfy all mathematical properties of cumulants



$$\mathrm{AC}_{2,1}(m,n) = \langle v_m^4 v_n^2\rangle - \langle v_m^4\rangle\langle v_n^2\rangle - 2\langle v_m^2 v_n^2\rangle\langle v_m^2\rangle + 2\langle v_m^2\rangle^2\langle v_n^2\rangle,$$

$$\begin{aligned}\mathrm{AC}_{3,1}(m,n) = {}& \langle v_m^6 v_n^2\rangle - \langle v_m^6\rangle\langle v_n^2\rangle - 3\langle v_m^2 v_n^2\rangle\langle v_m^4\rangle - 3\langle v_m^4 v_n^2\rangle\langle v_m^2\rangle \\ & + 6\langle v_m^4\rangle\langle v_m^2\rangle\langle v_n^2\rangle + 6\langle v_m^2 v_n^2\rangle\langle v_m^2\rangle^2 - 6\langle v_m^2\rangle^3\langle v_n^2\rangle,\end{aligned}$$

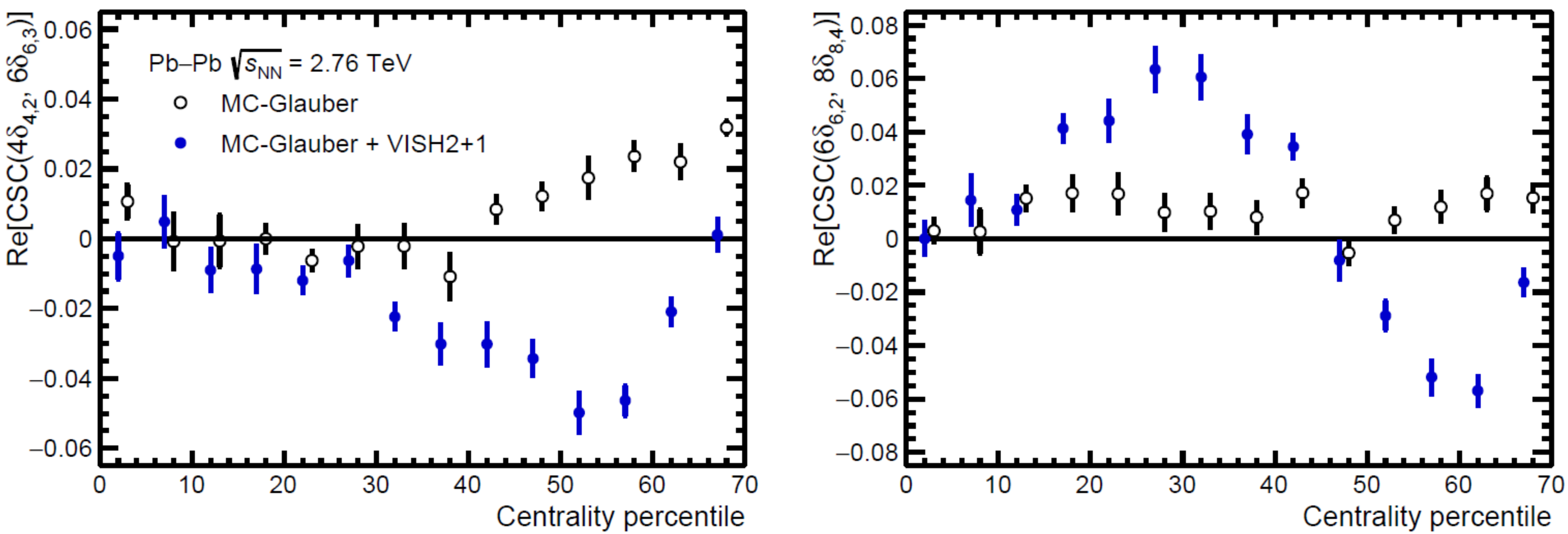
$$\begin{aligned}\mathrm{AC}_{4,1}(m,n) = {}& \langle v_m^8 v_n^2\rangle - \langle v_m^8\rangle\langle v_n^2\rangle - 4\langle v_m^2 v_n^2\rangle\langle v_m^6\rangle - 6\langle v_m^4 v_n^2\rangle\langle v_m^4\rangle \\ & + 6\langle v_m^4\rangle^2\langle v_n^2\rangle - 4\langle v_m^6 v_n^2\rangle\langle v_m^2\rangle + 8\langle v_m^6\rangle\langle v_m^2\rangle\langle v_n^2\rangle \\ & + 24\langle v_m^2 v_n^2\rangle\langle v_m^4\rangle\langle v_m^2\rangle + 12\langle v_m^4 v_n^2\rangle\langle v_m^2\rangle^2 \\ & - 36\langle v_m^4\rangle\langle v_m^2\rangle^2\langle v_n^2\rangle - 24\langle v_m^2 v_n^2\rangle\langle v_m^2\rangle^3 + 24\langle v_m^2\rangle^4\langle v_n^2\rangle,\end{aligned}$$

$$\begin{aligned}\mathrm{AC}_{2,1,1}(k,l,m) = {}& \langle v_k^4 v_l^2 v_m^2\rangle - \langle v_k^4 v_l^2\rangle\langle v_m^2\rangle - \langle v_k^4 v_m^2\rangle\langle v_l^2\rangle - \langle v_k^4\rangle\langle v_l^2 v_m^2\rangle \\ & + 2\langle v_k^4\rangle\langle v_l^2\rangle\langle v_m^2\rangle - 2\langle v_k^2 v_l^2\rangle\langle v_k^2 v_m^2\rangle - 2\langle v_k^2 v_l^2 v_m^2\rangle\langle v_k^2\rangle \\ & + 4\langle v_k^2 v_l^2\rangle\langle v_k^2\rangle\langle v_m^2\rangle + 4\langle v_k^2 v_m^2\rangle\langle v_k^2\rangle\langle v_l^2\rangle \\ & + 2\langle v_k^2\rangle^2\langle v_l^2 v_m^2\rangle - 6\langle v_k^2\rangle^2\langle v_l^2\rangle\langle v_m^2\rangle.\end{aligned}$$

Sec. V in arXiv:2101.05619

Cumulants of symmetry plane correlations

- By far the most difficult case to crack...
- These observables bring all previous measurements of symmetry plane correlations to the next level



Sec. VI in arXiv:2101.05619

‘Event-by-event cumulants of azimuthal angles’

A. Bilandzic, arXiv:2106.05760 (prepared for **‘Offshell-2021’**)

Main idea

- Traditional approach: cumulants of azimuthal angles are defined in terms of all-event averages:

$$c_n\{2\} \equiv \langle\langle e^{in(\varphi_1-\varphi_2)}\rangle\rangle - \langle\langle e^{in\varphi_1}\rangle\rangle\langle\langle e^{-in\varphi_2}\rangle\rangle$$

  - Due to underlying symmetries, all terms which are not isotropic are averaged out to 0 => fundamental properties of cumulants are lost

- New approach: cumulants of azimuthal angles are defined in terms of single-event averages:

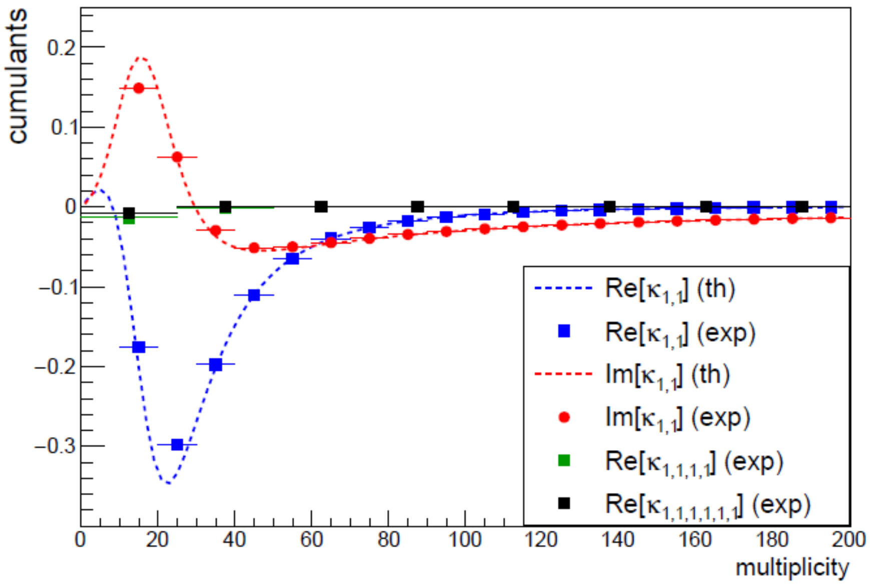
$$\kappa_{11} \equiv \langle e^{in(\varphi_1-\varphi_2)}\rangle - \langle e^{in\varphi_1}\rangle\langle e^{-in\varphi_2}\rangle$$

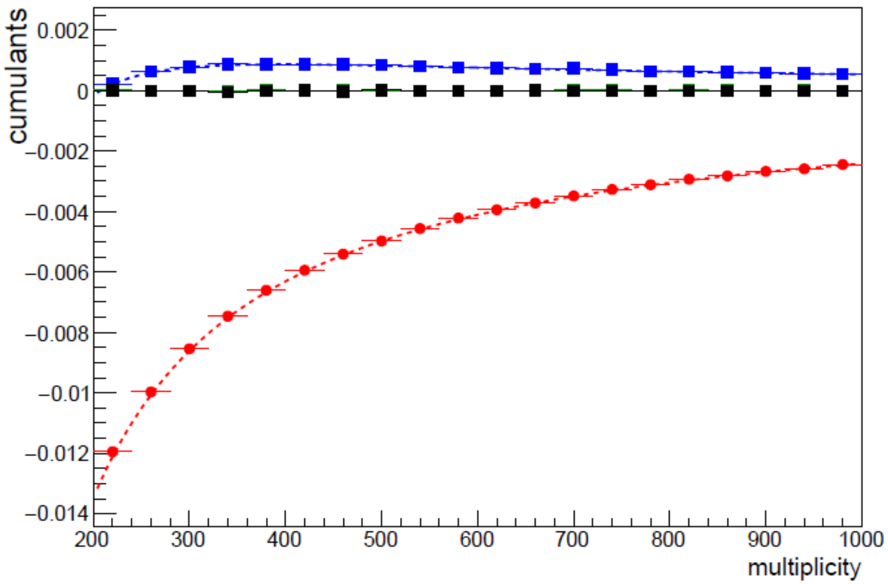
'Event-by-event cumulants of azimuthal angles'

  - Despite underlying symmetries, all terms are kept, and this remains true at all higher orders => interpretation and meaning of these new cumulants is completely different

Event-by-event cumulants of azimuthal angles

- Toy Monte Carlo study: Azimuthal angles are sampled pair-wise => only 2-particle correlations are present
  - New 2-particle cumulants correctly recover the theoretical input
  - New 4- and 6-particle cumulants are identically 0





- Works only if we have full control over combinatorial background

Sec. IV in arXiv:2101.05619

Role of combinatorial background

European Research Council
Established by the European Commission

- The origin of the problem: The dataset is randomized
  - Particles emitted in the same process: ‘signal’
  - Particles taken from different processes: ‘background’
- In most analyses in high-energy physics, ‘signal’ and ‘background’ are separated by using mixed-event technique
  - Not applicable for azimuthal angles, due to random event-by-event fluctuations of impact parameter vector
- Can we instead analytically solve the problem of combinatorial background?

TUM

Statistical independence

- If two random observables, x and y, are statistically independent, then their joined 2-particle probability density function (p.d.f.) fully factorizes into **marginal p.d.f.'s**:

$$f_{xy}(x,y) = f_x(x) f_y(y)$$

- Two marginal p.d.f.'s are defined as:

$$f_x(x) \equiv \int_Y f_{xy}(x,y)\, dy$$
$$f_y(y) \equiv \int_X f_{xy}(x,y)\, dx$$

- In general, $f_x(x)$ and $f_y(y)$ are two different p.d.f.'s

Three-particle correlations

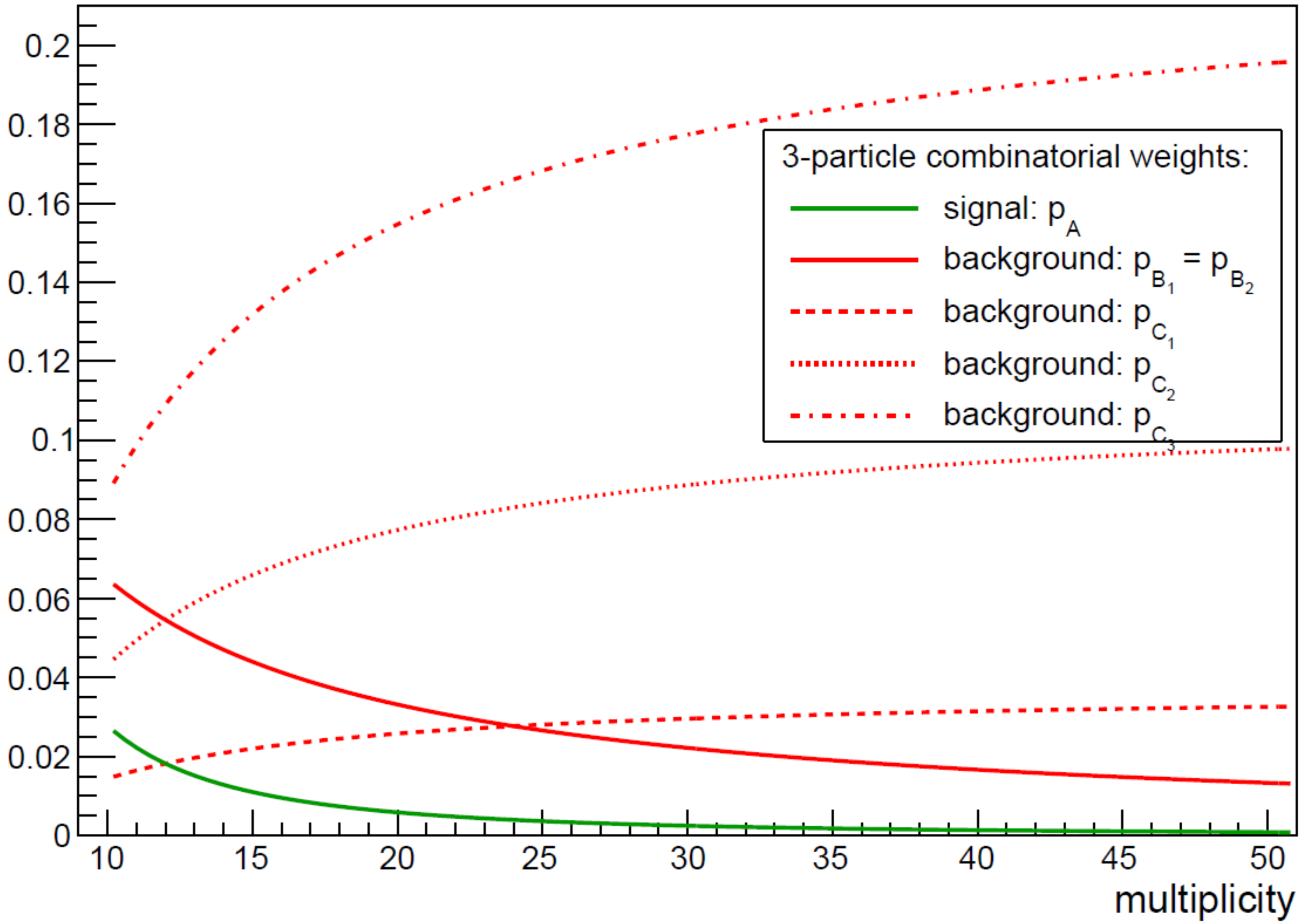
- If particles are emitted from p.d.f. $f(x,y,z)$, and if the resulting sample is randomized, what is the p.d.f. $w(x,y,z)$ which describes the final randomized sample?
- The most general result:

$$
\begin{aligned}
w(x,y,z) = {} & p_A f_{xyz}(x,y,z) \\
& + p_{B_1}\left[f_{xy}(x,y)f_x(z) + f_{xy}(x,y)f_y(z) + f_{xz}(x,z)f_x(y)\right. \\
& \quad \left. + f_{xz}(x,z)f_z(y) + f_{yz}(y,z)f_y(x) + f_{yz}(y,z)f_z(x)\right] \\
& + p_{B_2}\left[f_{xy}(x,y)f_z(z) + f_{xz}(x,z)f_y(y) + f_{yz}(y,z)f_x(x)\right] \\
& + p_{C_1}\left[f_x(x)f_x(y)f_x(z) + f_y(x)f_y(y)f_y(z) + f_z(x)f_z(y)f_z(z)\right] \\
& + p_{C_2}\left[f_x(x)f_x(z)f_y(y) + f_x(x)f_x(y)f_z(z) + f_y(y)f_y(z)f_x(x)\right. \\
& \quad \left. + f_y(y)f_y(x)f_z(z) + f_z(z)f_z(y)f_x(x) + f_z(z)f_z(x)f_y(y)\right] \\
& + p_{C_3} f_x(x)f_y(y)f_z(z)\,.
\end{aligned}
$$

- Universal combinatorial weights: $p_A, p_{B1}, p_{B2}, p_{C1}, p_{C2}, p_{C3}$
- Marginal p.d.f.'s: $f_x(x), f_y(y), f_z(z), f_{xy}(x,y), f_{xz}(x,z), f_{yz}(y,z)$

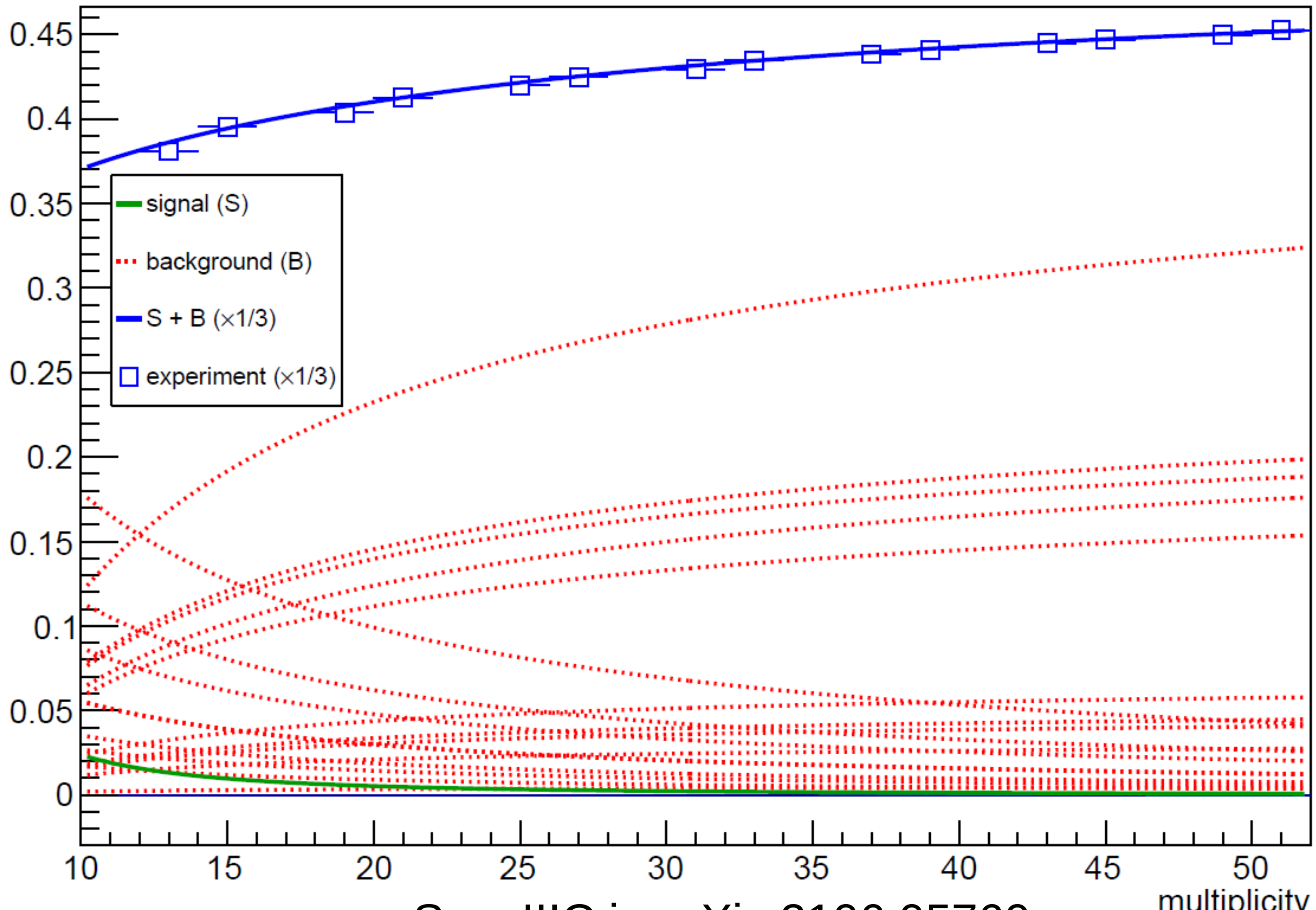
Combinatorial weights (3-particle)

- Universal and depend only on multiplicity:



Toy Monte Carlo (3-particle)

- Quantitative description of 3-particle azimuthal correlation in the randomized sample



Sec. IIIC in arXiv:2106.05760

Thanks!

Backup slides

Role of symmetries

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Established by the European Commission

- Cumulant is identically 0 if one of the variables in it is statistically independent of the others
    - This holds true over the whole phase space
- Reflection symmetry
    - Cumulant can be accidentally 0 due to symmetry $f(x,y) = f(x,-y)$ but in this case they are never 0 over the whole phase space
- Permutation symmetry
    - Marginal distributions of different variables are the same
- Frame independence
- Relabelling
    - Azimuthal correlators of different variables are estimated from exactly the same sample => properties of cumulants are lost

Sec. II in arXiv:2106.05760

TUM

Two-particle correlations

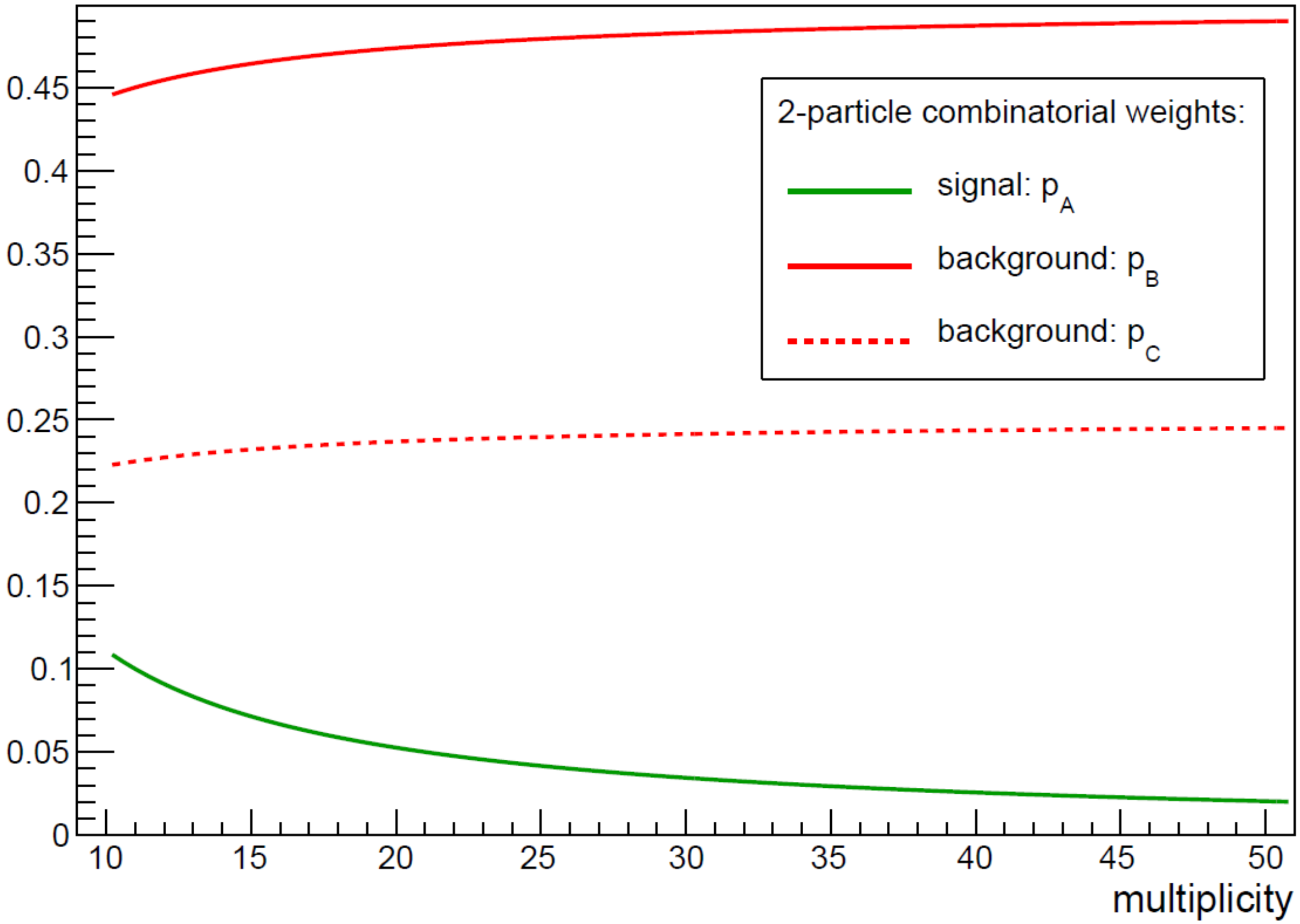
- If particles are emitted from p.d.f. $f(x,y)$, and if the resulting sample is randomized, what is the p.d.f. $w(x,y)$ which describes the final randomized sample?
- The most general result:

$$w(x,y) = p_A f_{xy}(x,y) + p_B f_x(x) f_y(y) + p_C \left[f_x(x) f_x(y) + f_y(x) f_y(y) \right]$$

- Universal combinatorial weights: p_A, p_B, p_C
- Marginal p.d.f.'s: $f_x(x)$, $f_y(y)$

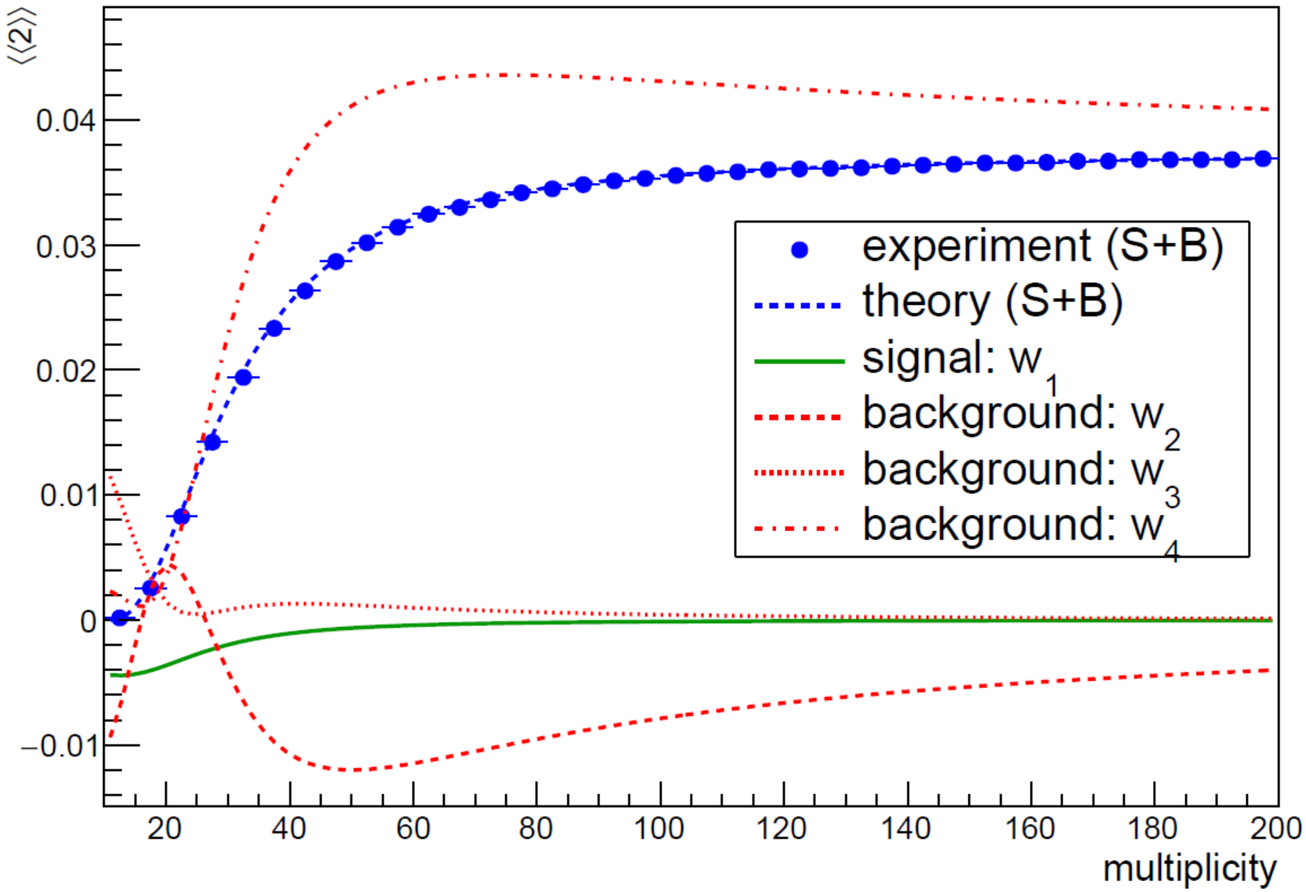
Combinatorial weights (2-particle)

- Universal and depend only on multiplicity:



Toy Monte Carlo (2-particle)

- Quantitative description of 2-particle azimuthal correlation in the randomized sample



Sec. IIIB in arXiv:2106.05760

Example: 2-particle cumulants

- How to use this new recipe in practice?
- Reminder: General 2-particle cumulant

$$\langle X_1 X_2 \rangle_c = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$$

As an elementary example, we perform these two checks for the simplest two-variate cumulant, $\kappa(X_1, X_2) = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$. The first check leads immediately to $\kappa(X_1, X_2) = \langle X_1 \rangle \langle X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle = 0$. Following the second check, we have that $\kappa(X) = \langle X^2 \rangle - \langle X \rangle^2$, so that:

$$\begin{aligned}\kappa(aX+b) &= \left\langle (aX+b)^2 \right\rangle - \langle aX+b \rangle^2 \\ &= a^2 \left\langle X^2 \right\rangle + 2ab \langle X \rangle + b^2 - a^2 \langle X \rangle^2 - 2ab \langle X \rangle - b^2 \\ &= a^2 \left(\left\langle X^2 \right\rangle - \langle X \rangle^2 \right) \\ &= a^2 \kappa(X)\,, \end{aligned} \tag{24}$$

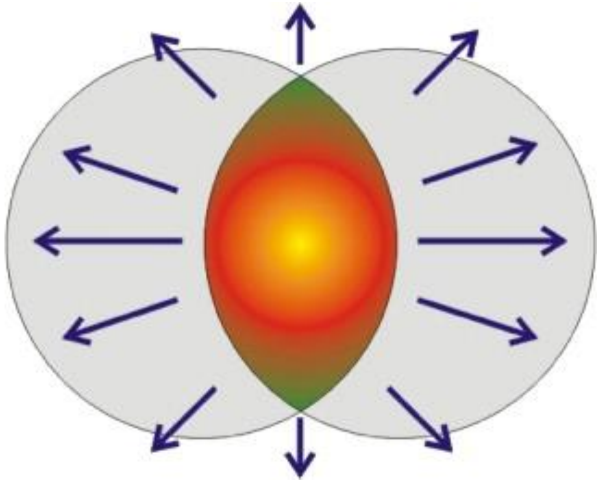
as it should be for a two-variate cumulant.

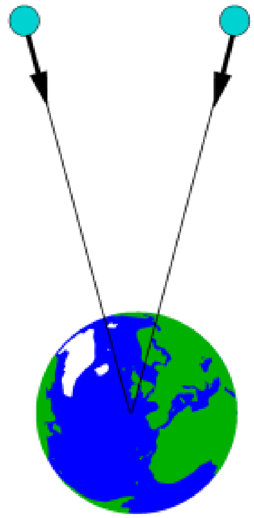
- Despite its simplicity, most of observables named cumulants in the field fail to satisfy this new recipe. What are the alternatives?

AB, M. Lesch, C. Mordasini, F. Taghavi, arXiv:2101.05619

The ‘flow principle’

- Correlations among all produced particles are induced solely by correlation of each single particle to the collision geometry





- Analogy with the falling bodies in gravitational field (rhs)
- Whether or not particle are emitted simultaneously, or one by one, trajectories are the same
  - These are **statistically independent** trajectories

Statistical independence, back to flow

- If anisotropic flow is the only source of correlations between produced particles, their joint n-variate p.d.f.

$$f(\varphi_1,\ldots,\varphi_n)$$

factorizes into product of n single-particle marginal p.d.f.'s:

$$f(\varphi_1,\ldots,\varphi_n) = f_{\varphi_1}(\varphi_1)\cdots f_{\varphi_n}(\varphi_n)$$

- From ‘flow principle’: All marginal p.d.f.’s are the same, and therefore parameterized by the same Fourier series:

$$\begin{aligned} f(\varphi_1,\ldots,\varphi_n) &= f(\varphi_1)\cdots f(\varphi_n) \\ f(\varphi) &= \frac{1}{2\pi}\left[1+2\sum_{n=1}^{\infty} v_n\cos[n(\varphi-\Psi_n)]\right] \end{aligned}$$

Elementary example: 2-particle correlation

- When only flow correlations are present, the relation between azimuthal correlators and flow moments is exact!
  - For instance:

$$\langle\cos[n(\varphi_1-\varphi_2)]\rangle = v_n^2$$

- This can be derived analytically in merely 3 steps:

$$\langle\cos[n(\varphi_1-\varphi_2)]\rangle \equiv \int_0^{2\pi}\int_0^{2\pi}\cos[n(\varphi_1-\varphi_2)]f(\varphi_1,\varphi_2)\,d\varphi_1 d\varphi_2$$

- Then, use:
  1. Factorization of joint p.d.f. $f(\varphi_1,\varphi_2) = f(\varphi_1)\, f(\varphi_2)$
  2. Each single particle p.d.f. $f(\varphi)$ is given by same Fourier series
  3. Orthogonality relations of trigonometric functions

- Exactly the same derivation works for any other correlator!

Q-vectors

- Q-vectors (or flow vectors) are among the most important fundamental objects in flow analyses nowadays
- Three definitions:
  - M-particle Q-vector

$$Q_n \equiv \sum_{i=1}^{M} e^{in\varphi_i}$$

  - Unit Q-vector

$$u_n \equiv e^{in\varphi}$$

  - Reduced Q-vector

$$q_n \equiv \frac{Q_n}{\sqrt{M}}$$

Q-vectors

- What Q-vectors have to do with multi-particle correlation techniques?
- Remarkaby, we can **analytically express any multi-particle azimuthal correlator in terms of Q-vectors** in such a way that all self-correlations are exactly removed
  - First realized by S. Voloshin ~ 10 years ago
  - This realization is the most important breaktrough in the field of correlation techniques of late
- Example:

$$\begin{aligned}
\langle 2\rangle &\equiv \langle\cos(n(\varphi_1-\varphi_2))\rangle\\
&\equiv \frac{1}{\binom{M}{2}2!}\sum_{\substack{i,j=1\\(i\neq j)}}^{M} e^{in(\varphi_i-\varphi_j)}\\
&= \frac{1}{\binom{M}{2}2!}\times\left[|Q_n|^2-M\right]
\end{aligned}$$

Q-vectors

- Example: Analytic result for 4-p correlation

$$Q_n = \sum_{i=1}^{M} e^{in\varphi_i}$$

$$Q_{2n} = \sum_{i=1}^{M} e^{i2n\varphi_i}$$

$$\begin{aligned}
\langle 4 \rangle &\equiv \langle \cos(n(\varphi_1+\varphi_2-\varphi_3-\varphi_4)) \rangle \\
&= \frac{1}{\binom{M}{4}4!} \sum_{\substack{i,j,k,l=1 \\ (i\neq j\neq k\neq l)}}^{M} e^{in(\varphi_i+\varphi_j-\varphi_k-\varphi_l)} \\
&= \frac{1}{\binom{M}{4}4!} \times \left[|Q_n|^4 + |Q_{2n}|^2 - 2\cdot\mathfrak{Re}\left[Q_{2n}Q_n^*Q_n^*\right] - 4(M-2)\,|Q_n|^2 \right. \\
&\quad \left. + 2M(M-3) \right]
\end{aligned}$$

- The key point: The RHS can be obtained in the single loop over all azimuthal angles of particles
  - Both exact and fast formalism

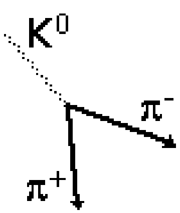
A.B. *et al,* Phys. Rev. C **89** (2014) no.6, 064904 [arXiv:1312.3572 [nucl-ex]].

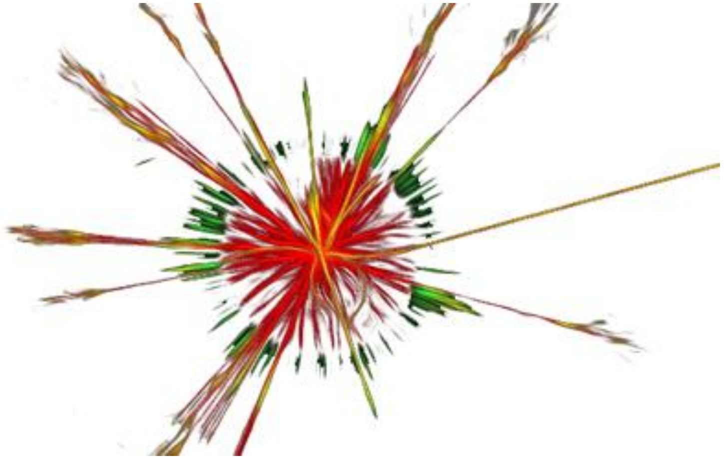
Nonflow

- All other sources of contributions to azimuthal correlators, besides flow correlations, we classify as **nonflow**
  - Due to nonflow, flow degrees of freedom estimated with azimuthal correlators, will be systematically biased
- It is hopeless to quantify all possible sources of nonflow
  - Is there a systematic way to suppress ‘em all?
- Flow vs. nonflow:
  - Flow is collective effect, correlates all particles
  - Nonflow is generally a correlation among few particles

Nonflow examples

- **Physical:** Resonance decays, jets, etc.
- **Detector artifacts:** Track splitting in reconstruction, etc.
- **Computational:** Autocorrelations



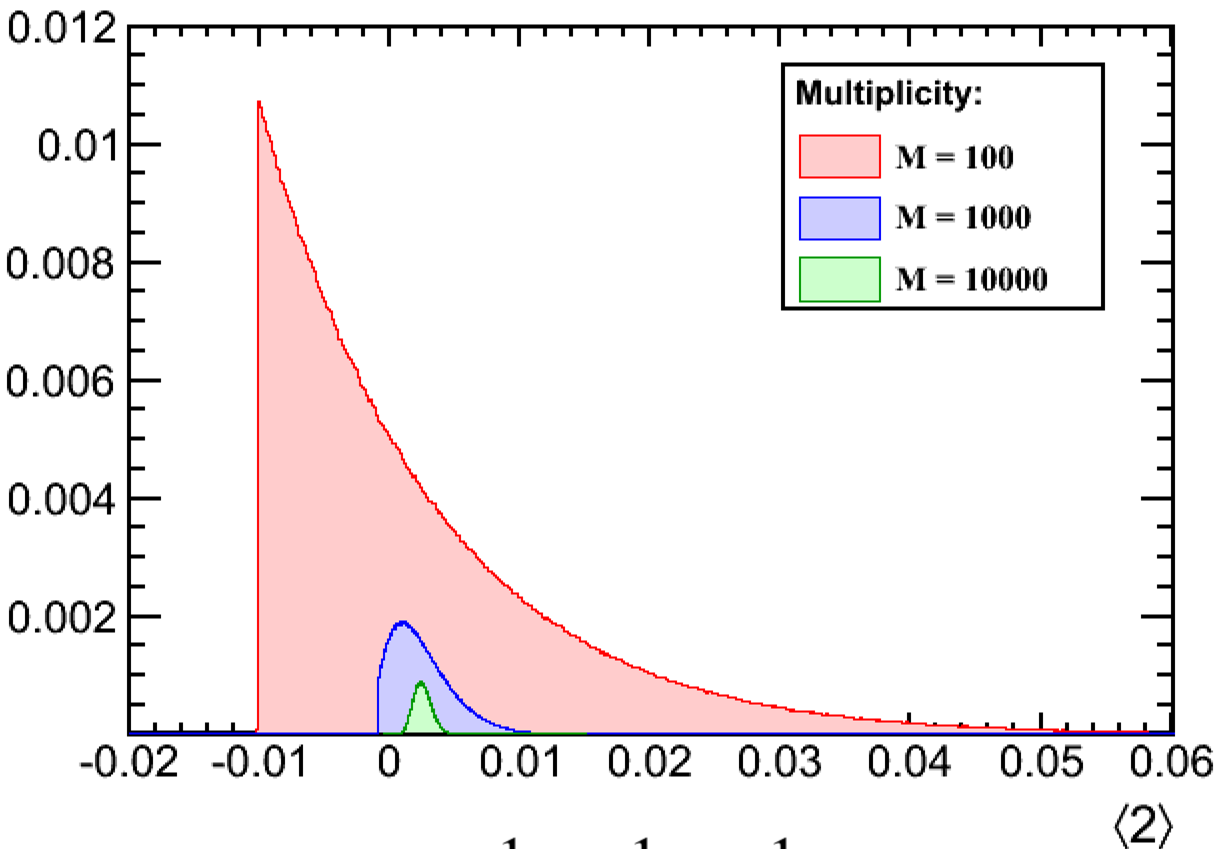


$$\left\langle e^{in(\varphi_1-\varphi_2)}\right\rangle, \quad \varphi_1 \neq \varphi_2$$

$$\left\langle e^{in(\varphi_1+\varphi_2-\varphi_3-\varphi_4)}\right\rangle, \quad \varphi_1 \neq \varphi_2 \neq \varphi_3 \neq \varphi_4$$

Multiparticle correlation techniques

- Monte Carlo study, fixed *v* = 0.05 as an input:



$$\sigma_v \sim \frac{1}{\sqrt{N}} \frac{1}{M^{k/2}} \frac{1}{v^{k-1}}$$